1 Kholshevnikov Metric

Consider two points in 3-dimensional space, defined by their radius vectors $\vec{r}_1$ and $\vec{r}_2$. According to the Pythagorean theorem, the distance between these points $d_{12} = \sqrt{(\vec{r}_1 - \vec{r}_2)^2}$. The concept of "distance" can be generalized to arbitrary spaces.

In order to determine which meteor shower a meteoroid belongs to, it is required to determine the "distance" between the orbits — the so-called *metric* in the space of orbital parameters. One example of metric is Kholshevnikov metric, described in detail below.

Let a, e, i, ω, Ω be the semi-major axis, eccentricity, inclination, argument of the pericentre and the longitude of the ascending node of the meteoroid's orbit, respectively (Keplerian elements). We introduce two vectors $\vec{u}$ and $\vec{v}$:

$$\vec{u} := \begin{pmatrix} \sin i\sqrt{a(1-e^2)}\sin\Omega \\ -\sin i\sqrt{a(1-e^2)}\cos\Omega \\ \cos i\sqrt{a(1-e^2)} \end{pmatrix}, \quad \vec{v} := \begin{pmatrix} e\sqrt{a(1-e^2)}(\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega) \\ e\sqrt{a(1-e^2)}(\cos\omega\sin\Omega + \cos i\sin\omega\cos\Omega) \\ e\sin i\sqrt{a(1-e^2)}\sin\omega \end{pmatrix},$$

Let us define a distance ρ — Kholshevnikov metric — between two orbits in the space of Keplerian elements by the formula

$$\rho = \sqrt{(\vec{u}_1 - \vec{u}_2)^2 + (\vec{v}_1 - \vec{v}_2)^2}.$$

Let a_0, e_0, i_0, ω_0, Ω_0 be the Keplerian elements of the mean orbit of the meteor shower. If the distance ρ between this orbit and the orbit of a given meteoroid is smaller than a limiting value, we may suppose the meteoroid to belong to this meteor shower.

The file `catalog2018.csv` contains the observation times and equatorial coordinates of meteoroids, as well as the Keplerian elements of the orbit of each meteoroid. The data was taken from the archives of the project SonotaCo Network.

Year	Month	Day	α, °	δ, °	a, au	e	i, °	ω, °	Ω, °
2018	1	1.42418	83.56900	1.11400	1.967	0.57950	8.81110	55.40330	100.69660
2018	1	1.48810	107.81900	24.12900	1.973	0.75290	1.44970	280.02610	280.76860
...	...	...	...	...	...	...	...	...	...

We are looking for meteoroids related to the Geminids meteor shower. The mean orbit of the shower corresponds to $a_0 = 1.31$ au, $e_0 = 0.889$, $i_0 = 22.9°$, $\omega_0 = 324.3°$, $\Omega_0 = 261.7°$. For this problem, let us set the limiting value $\log_{10}\rho = -1.0$.

a) Plot all meteoroids on the graph in the coordinates $(\lambda; \beta)$ where λ and β are the ecliptic longitude and latitude of the meteoroid, respectively.

b) How many meteoroids belong to the Geminids meteor shower? Use the limiting value of the $\log_{10}\rho$ specified above and consider only meteoroids with elliptic orbits. Mark these meteoroids on the graph in the coordinates $(\lambda; \beta)$.

c) Plot histograms for semi-major axes and $\log_{10}\rho$ values for meteoroids belonging to the Geminids meteor shower and the histogram for days of observing these meteoroids. On what date was the largest number of meteoroids from the Geminids meteor shower observed?

d) Determine the median ecliptic and equatorial coordinates of the meteoroids belonging to the Geminids meteor shower.

In memory of ***Konstantin Vladislavovich Kholshevnikov (1939–2021),*** *head of the Department of Celestial Mechanics at St. Petersburg State University, honored scientist of the Russian Federation, whose name was given to the minor planet (3504) Kholshevnikov.*

Solution:

a) First, we convert the equatorial coordinates $(\alpha;\delta)$ into ecliptic coordinates $(\lambda;\beta)$ by writing spherical trigonometry formulae for a triangle with vertices at the north celestial pole, the north ecliptic pole and a meteoroid:

$$\begin{aligned}
\sin\beta &= \sin\delta\cos\varepsilon - \cos\delta\sin\varepsilon\sin\alpha,\\
\cos\beta\cos\lambda &= \cos\delta\cos\alpha,\\
\cos\beta\sin\lambda &= \sin\delta\sin\varepsilon + \cos\delta\cos\varepsilon\sin\alpha,
\end{aligned}$$

where $\varepsilon = 23^\circ\,26'$ is the inclination of the celestial equator to the ecliptic.

Numerous clumps are visible in the Fig. 1. These are what meteor showers look like if projected on the celestial sphere.

b) We denote the vectors $\vec{u}$ and $\vec{v}$ for the mean orbit of the meteor shower as $\vec{u}_0$, $\vec{v}_0$.

$$\vec{u}_0 = \begin{pmatrix} -0.2018 \\ 0.0294 \\ 0.4828 \end{pmatrix}, \quad \vec{v}_0 = \begin{pmatrix} -0.3025 \\ -0.3383 \\ -0.1058 \end{pmatrix}.$$

The problem condition states that we should consider only meteoroids in elliptical orbits, so objects with an orbital eccentricity greater than or equal to 1 are immediately excluded from consideration. For each meteoroid remaining in the catalog, we calculate the vectors $\vec{u}$ and $\vec{v}$ and determine the value of ρ. Next, we leave only objects with $\log_{10}\rho \leqslant -1.0$. There are only 2742 such objects.

When calculating the metric, we substitute the value of semi-major axis in au since this unit of measurement is natural for the problem in consideration. $\log_{10}\rho$ values for all meteoroids from the catalog are shown in Fig. 2. The blue area corresponds to the meteoroids from the Geminids meteor shower. Meteoroids from the Geminids meteor shower are marked in red on the distribution of all meteoroids in Fig. 3.

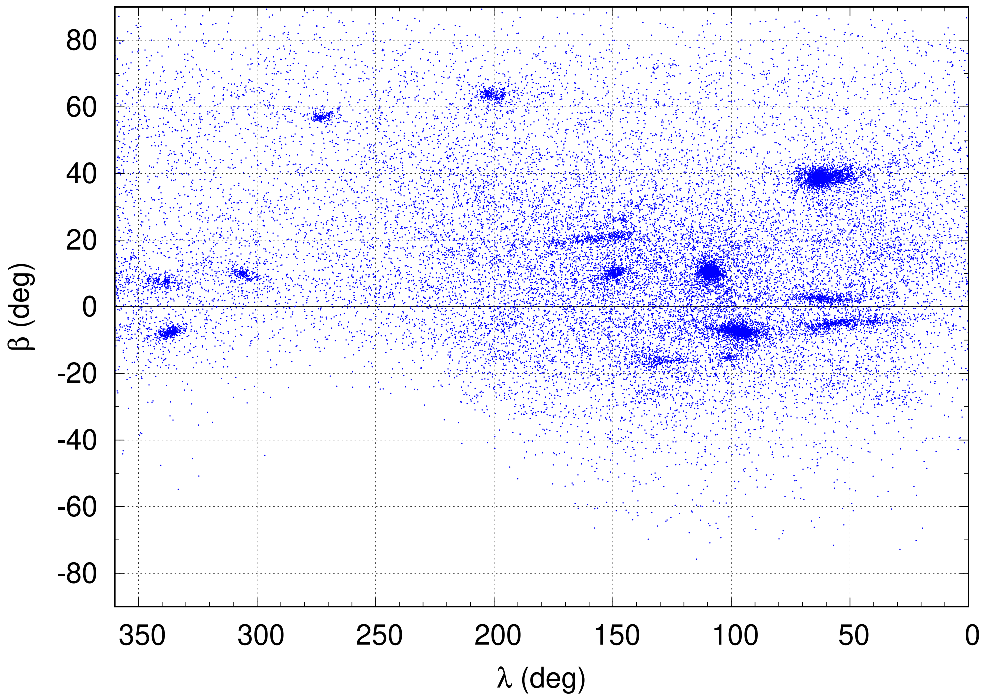


Figure 1: Distribution of the meteoroids on the celestial sphere in ecliptic coordinates

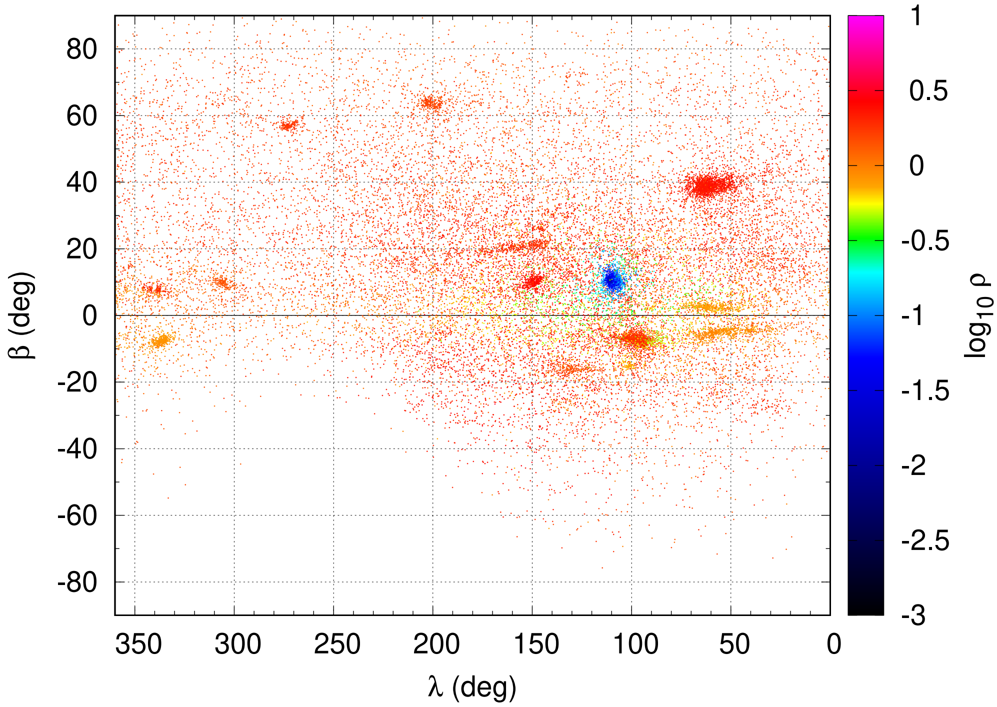


Figure 2: $\log_{10}\rho$ values for all meteoroids from the catalog

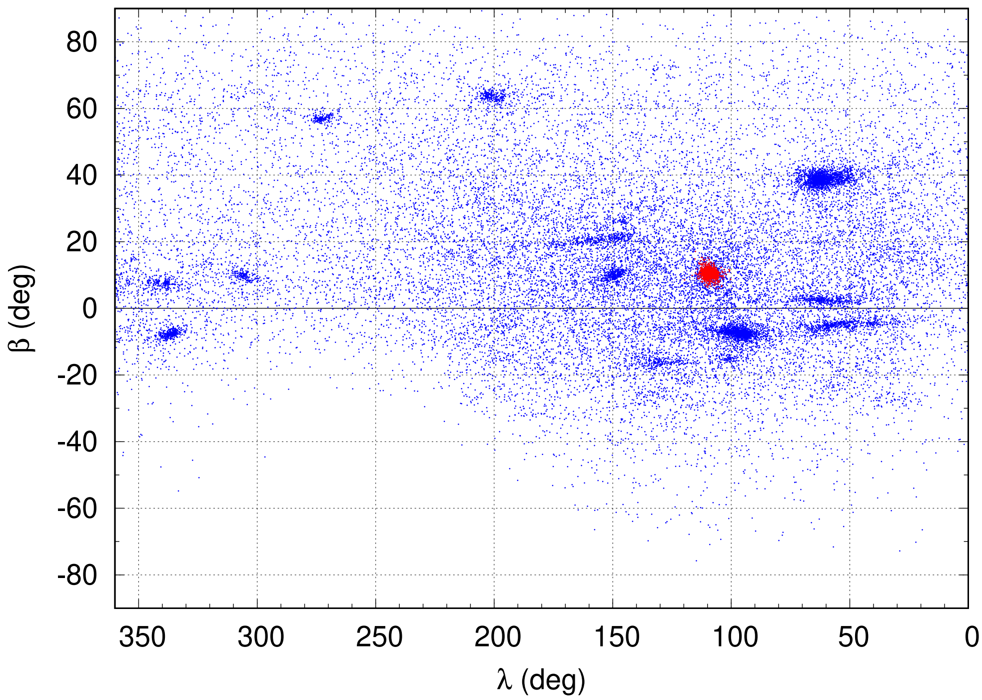


Figure 3: Geminids are red, other meteoroids are blue, if you are an astronomer, we love you!

c) The distribution of the semi-major axes of the orbit turns out to be quite wide. Objects with a semi-major axis up to 4 au are observed. Thus, the criterion for selecting objects by metric only is not absolutely accurate. However, most objects outline a narrow distribution peak.

All meteoroids that fit the criterion were observed in December, therefore, along the vertical axis on the corresponding histogram, we mark the days of December. The largest number of meteoroids was observed on December 14. In reality, the meteor shower is active from December 4 to December 17 and the maximum activity really falls on December 14.

d) In order to find the median values of the coordinates, we must separately sort each coordinate in ascending order and take the central value (or half-sum of the central values) of the resulting ordered array:

$$\mathsf{Me}\,\delta = 32.37^\circ, \quad \mathsf{Me}\,\alpha = 113.29^\circ;$$
$$\mathsf{Me}\,\beta = 10.76^\circ, \quad \mathsf{Me}\,\lambda = 109.94^\circ.$$

Note that the median equatorial coordinates are in good agreement with the accepted coordinates of the radiant, $\delta_0 = 32.3^\circ$, $\alpha_0 = 113.5^\circ$ (Jenniskens et al., 2016, Icarus, V. 266, p. 331).

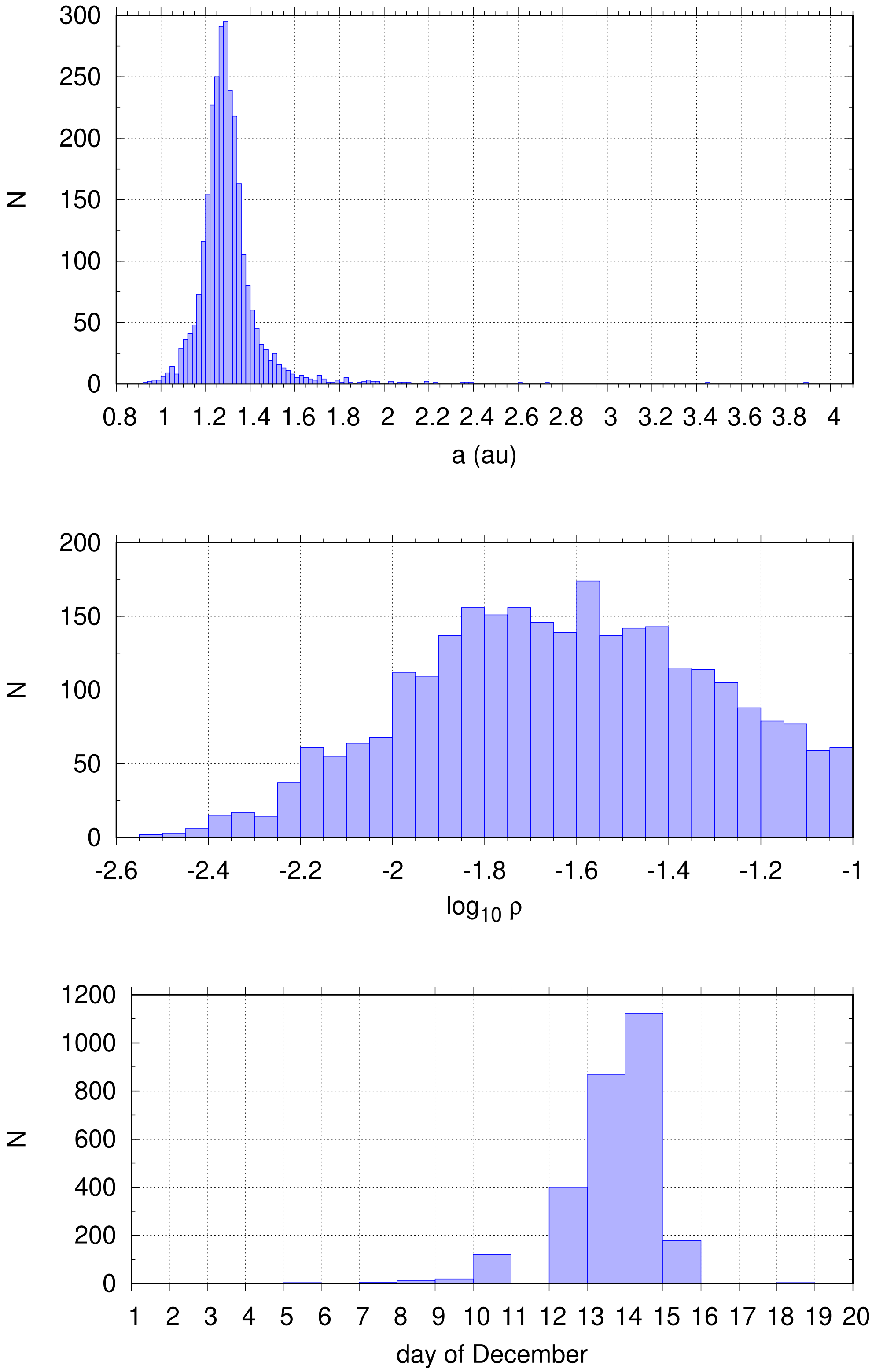


Figure 4: Distributions of some parameters of observed meteoroids

Marking Scheme:

- $(\alpha; \delta) \rightarrow (\beta; \lambda)$ — **3 pt.**
- Distribution of the meteoroids on the celestial sphere — **2 pt.**
- $\log_{10} \rho$ calculation — **3 pt.**
- Geminids highlighted — **2 pt.**
- Distribution of parameters (histograms) — **2 pt.** × **3**
- Estimation of median coordinates — **1 pt.** × **2** × **2**

2 Metallica

The file clusters.csv contains the parameters of several dozen open clusters of the Milky Way (Viscasillas Vázquez et al., 2023, arXiv:2309.17153). The columns indicate:

- the name of the cluster,
- metallicity [M/H] (dex),
- α-element enhancement [α/Fe] (dex),
- age (in billion years),
- galactocentric distance $R_{\rm GC}$ (in pc),
- orbital eccentricity e,
- maximum deviation from the Galactic plane $Z_{\max}$ (in kpc),
- velocity components in cylindrical galactocentric coordinate system (radial V_R, azimuthal V_ϕ, vertical V_Z, in km/s).

Cluster Name	[M/H] dex	[α/Fe] dex	Age Gyr	R_{GC} pc	e	$Z_{\max}$ kpc	V_R km/s	V_ϕ km/s	V_Z km/s
Alessi1	−0.052	−0.029	1.445	8637.1	0.063	0.23	17.29	230.996	12.241
Berkeley89	0.047	−0.024	2.089	8473.7	0.173	0.338	28.563	205.102	8.261
...	...	...	...	...	...	...	...	...	...

Here "dex" is a contraction of "decimal exponent", a convenient unit indicating any number or ratio's order-of-magnitude. For example, 100 could be described as 2 dex, or two numbers that differ by a factor of 1000 could be said to differ by 3 dex.

In the cylindrical galactocentric coordinate system,

- the radial velocity V_R is measured along the galactic axial distance,
- the azimuthal velocity V_ϕ is measured in the direction of rotation of the Galaxy,
- the vertical velocity V_Z is measured perpendicular to the galactic disk.

a) What are α-elements? Give three examples.

b) How many times is the ratio of the concentration of metal atoms to the concentration of hydrogen atoms in the oldest cluster of the catalogue greater than in the Sun?

c) Plot a graph of [M/H] versus the α-element enhancement [α/Fe]. Determine the coefficient of linear correlation between these values. Explain the presence or absence of correlation (or anticorrelation). How will the correlation coefficient change when the object with the smallest [α/Fe] is excluded?

The coefficient r_{xy} of linear correlation between the values of x and y is determined by the formula

$$r_{xy} = \frac{\sum_{i=1}^{N} (x_i - \langle x\rangle)(y_i - \langle y\rangle)}{\sqrt{\sum_{i=1}^{N} (x_i - \langle x\rangle)^2}\sqrt{\sum_{i=1}^{N} (y_i - \langle y\rangle)^2}}\,; \quad \langle x\rangle = \frac{1}{N}\sum_{i=1}^{N} x_i\,, \quad \langle y\rangle = \frac{1}{N}\sum_{i=1}^{N} y_i\,.$$

d) Plot a graph of e versus the ratio V_p/V, where $\vec{V}_p$ is the velocity component perpendicular to $\vec{V}_\phi$ and V is the total velocity. Determine the coefficient of linear correlation between e and V_p/V. Explain the presence or absence of correlation (or anticorrelation).

Solution:

a) Alpha-element or *the alpha process element* is any of the elements that can be created by the alpha process, the combination of a nucleus and an alpha particle (alpha capture), which are several elements with even atomic numbers. The most abundant isotopes are integer multiples of four — the mass of the helium nucleus. The stable alpha elements are C, O, Ne, Mg, Si and S.

b) NGC 6791 is the oldest open cluster in the catalogue. Nevertheless, its metallicity is 0.136, so the cluster is one of the oldest and most metal-rich clusters in the Milky Way. By definition of metallicity, which is determined in relation to the Sun,

$$[\mathrm{M/H}] = \log_{10}\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right) - \log_{10}\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right)_\odot ,$$

$$0.136 = \log_{10}\left(\frac{\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right)}{\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right)_\odot}\right) \quad \Rightarrow \quad \frac{\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right)}{\left(\frac{n_\mathrm{M}}{n_\mathrm{H}}\right)_\odot} = 1.4.$$

c) The anticorrelation between the two values is noticeable on the graph, although the points are distributed quite widely.

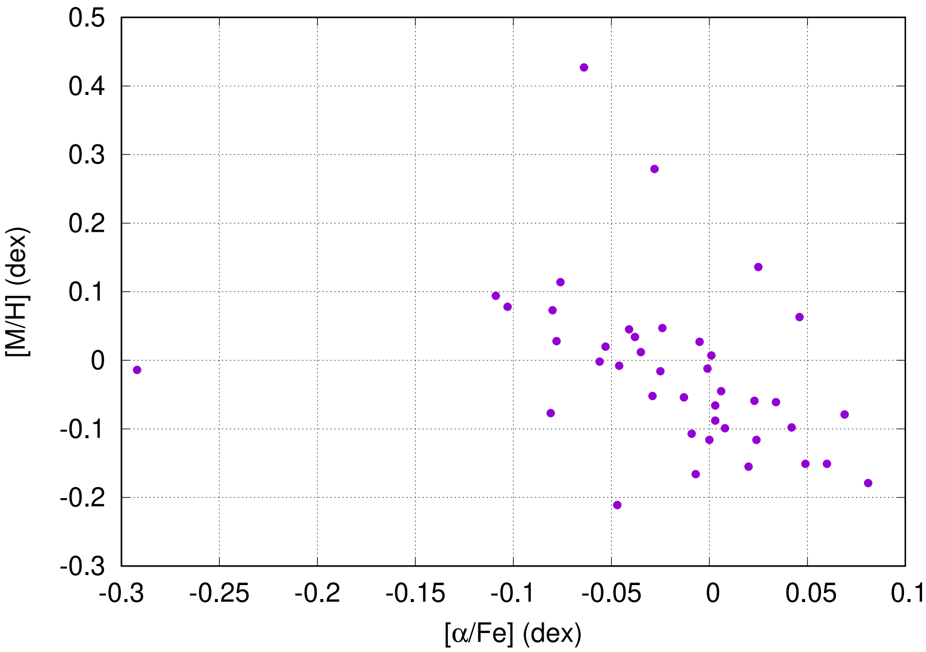


Figure 5: Scatter plot for the metallicity and α-element enhancement

To estimate the correlation coefficient, we have to calculate several additional values:

$$\langle[\mathrm{M/H}]\rangle = -0.017, \quad \langle[\alpha/\mathrm{Fe}]\rangle = -0.021,$$
$$\sum_{i=1}^{N}([\mathrm{M/H}] - \langle[\mathrm{M/H}]\rangle)([\alpha/\mathrm{Fe}] - \langle[\alpha/\mathrm{Fe}]\rangle) = -0.110,$$
$$\sum_{i=1}^{N}([\mathrm{M/H}] - \langle[\mathrm{M/H}]\rangle)^2 = 0.583, \qquad \sum_{i=1}^{N}([\alpha/\mathrm{Fe}] - \langle[\alpha/\mathrm{Fe}]\rangle)^2 = 0.162;$$
$$r = -0.358.$$

UPK 21 has the smallest value of $[\alpha/\mathrm{Fe}]$. It might be an outlier or a very young object. When this object is excluded, the following values are obtained:

$$\langle[\mathrm{M/H}]\rangle = -0.017, \qquad \langle[\alpha/\mathrm{Fe}]\rangle = -0.014,$$
$$\sum_{i=1}^{N}([\mathrm{M/H}] - \langle[\mathrm{M/H}]\rangle)([\alpha/\mathrm{Fe}] - \langle[\alpha/\mathrm{Fe}]\rangle) = -0.109,$$
$$\sum_{i=1}^{N}([\mathrm{M/H}] - \langle[\mathrm{M/H}]\rangle)^2 = 0.583, \qquad \sum_{i=1}^{N}([\alpha/\mathrm{Fe}] - \langle[\alpha/\mathrm{Fe}]\rangle)^2 = 0.086;$$
$$r = -0.487.$$

Theoretical galactic evolution models predict that early in the Universe there were more α-elements relative to iron. At higher metallicities the α-poor stars are interpreted as the thin disk. On the other hand, low metallicities and the high-α sequence is attributed to the halo and at high metallicities the high-α sequence is attributed to the thick disk although there is likely significant overlap between the two (see Hawkins et al., 2015, MNRAS, 458, 758H).

d)

$$\begin{cases} V_p = \sqrt{V_R^2 + V_z^2}, \\ V = \sqrt{V_R^2 + V_\phi^2 + V_z^2} \end{cases} \quad \Rightarrow \quad \frac{V_p}{V} = \sqrt{\frac{V_R^2 + V_z^2}{V_R^2 + V_\phi^2 + V_z^2}}.$$

The relationship between the values on the graph is clearly visible, and the explanation is quite simple. The inclination of most orbits is small, since the declination from the galactic plane is small compared to the galactocentric distance. Consequently, an increase in the fraction of velocity perpendicular to the azimuthal component leads to increased fluctuations in the galactocentric distance and to an increase in eccentricity. Nevertheless, the scatter of points relative to the median line is quite noticeable.

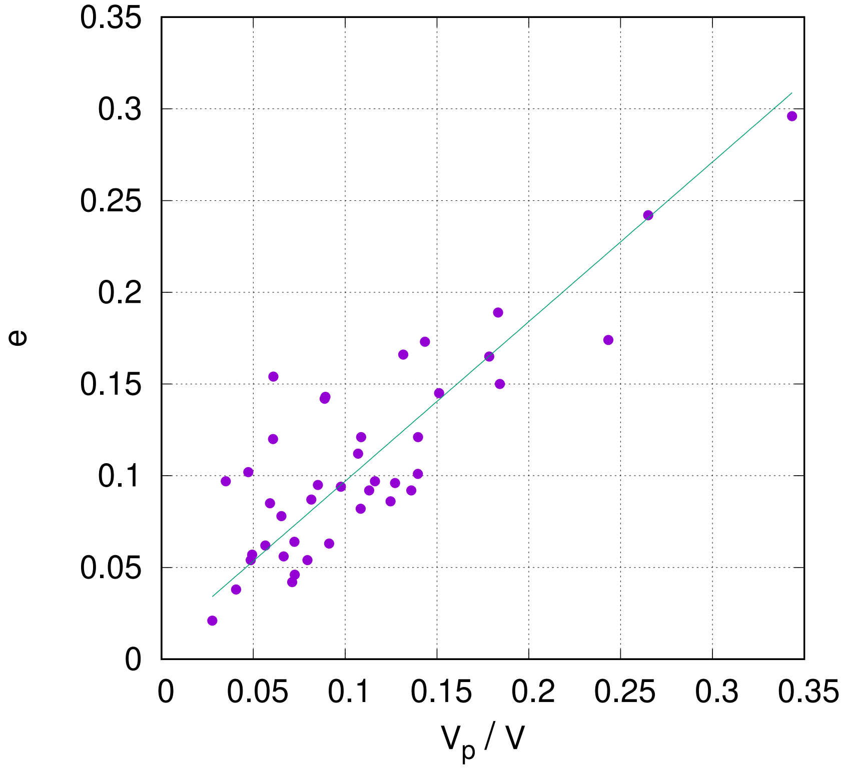


Figure 6: Scatter plot for the velocity ratio and the eccentricity

To estimate the correlation coefficient, we need to calculate several additional values:

$$\left\langle \frac{V_p}{V} \right\rangle = 0.110, \qquad \langle e \rangle = 0.109,$$

$$\sum_{i=1}^{N} \left(\frac{V_p}{V} - \left\langle \frac{V_p}{V} \right\rangle \right) (e - \langle e \rangle) = 0.123,$$

$$\sum_{i=1}^{N} \left(\frac{V_p}{V} - \left\langle \frac{V_p}{V} \right\rangle \right)^2 = 0.169, \qquad \sum_{i=1}^{N} (e - \langle e \rangle)^2 = 0.125;$$

$$r = 0.846.$$

Marking Scheme:

- α-elements:
  - Explanation of the concept / definition — **1 pt.**
  - Examples — **1 pt.** $\times$ **3**
- Metallicity ratio on a linear scale — **2 pt.**
- [M/H] vs. [α/Fe]:
  - Plot — **2 pt.**
  - r — **3 pt.**
  - Anticorrelation explanation — **1 pt.**
  - Outlier correction — **2 pt.**
- e vs. V_p/V:
  - Plot — **2 pt.**
  - r — **3 pt.**
  - Correlation explanation — **1 pt.**

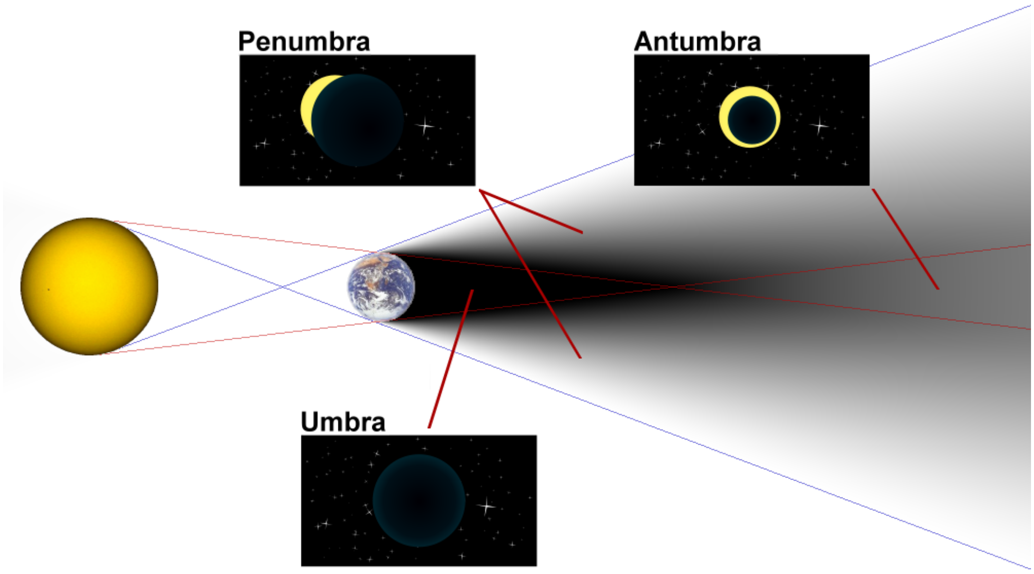
3 Eclipse in Siberia

There are two videos:

- eclipse1.mp4 or https://youtu.be/nsWVPedsQNk
- eclipse2.mp4 or https://youtu.be/Lw7GGm0U94g

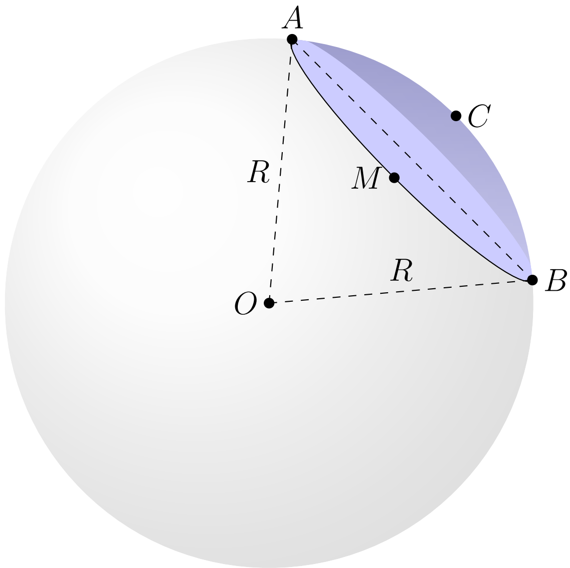
showing conditions for the annular solar eclipse on July 25, 2120. The greatest eclipse will be observed in Siberia, geographical region in Russia. Estimate the altitude of the Sun at the point where the maximum magnitude of the eclipse will be observed (if the weather is good).

Hint: the center of lunar antumbra reaching the Earth's surface is marked by green cross.



Solution:

One of the videos shows Earth and lunar shadow as it seen from some point of view located in the direction to the Sun. The center of antumbra goes from point A to point B. Maximal magnitude of eclipse is observed at point M, the closest to the Moon.



For observer in point C (also in A and B) the Sun is located on the horizon. The altitide of the Sun at point M is equal[1] to the angular measure of arc MC.

The duration of the eclipse is short, so we can assume that the shadow's velocity is equal to the orbital velocity of the Moon moving parallel to the picture plane:

$$v = \sqrt{\frac{G\mathfrak{M}_\oplus}{a_☾}} = \sqrt{\frac{(6.67 \cdot 10^{-11}) \times (5.97 \cdot 10^{24})}{3.84 \cdot 10^8}} \approx 1.0 \cdot 10^3 \text{ m/s} = 1 \text{ km/s}.$$

From the video, we can find that the center of the lunar shadow reaches the surface of the Earth at about 14:31 UT and leaves it at 14:47 UT. Thus, this path takes 16 minutes and the line segment

$$|AB| = vt = 1 \times (16 \cdot 60) = 960 \text{ km}.$$

Now we can determine the arcs:

$$|AB|^2 = 2R^2 - 2R^2 \cos\angle AOB,$$

$$\angle AOB = \arccos\left(1 - \frac{|AB|^2}{2R^2}\right) = \arccos\left(1 - \frac{960^2}{2 \cdot 6371^2}\right) = 8.6^\circ.$$

It is obvious that the arcs AC, BC and MC are equal due to symmetry, so

$$h_\odot = \frac{\angle AOB}{2} \approx 4^\circ.$$

[1]The horizontal parallax of the Sun is negligible.

Marking Scheme:

- Using the author's method:
  - Duration of shadow passage — **5 pt.**
  - Arc length in degrees — **5 pt.**
  - Geometry of the problem — **5 pt.**
  - Altitude of the Sun: evaluation + final result — **2 pt.** + **3 pt.**
- OR using the coordinates of the Sun:
  - Accounting for local time correction — **4 pt.**
  - Rotation is negligible during the annular phase — **2 pt.**
  - Coordinates of the shadow center at the beginning/middle of the eclipse — **4 pt.**
  - Declination of the Sun — **2 pt.**
  - Time to sunset/culmination — **3 pt.**
  - Altitude of the Sun: evaluation + final result — **2 pt.** + **3 pt.**
- OR using the position of the terminator *(max. 15 points):*
  - Distance to the terminator in km and degrees — **5 pt.**
  - Accounting for latitude distortions on the rectangular map — **5 pt.**
  - Altitude of the Sun: evaluation + final result — **2 pt.** + **3 pt.**

  This method is rough. The rectangular map projection is unknown.

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Hubble constant	$H_0 = 70$ (km/s)/Mpc
Astronomical unit	1 au = $149.6 \cdot 10^6$ km
Parsec	1 pc = 206 265 au

Earth

Radius	$R_\oplus = 6371$ km
Obliquity	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^\mathrm{d}$
Orbital eccentricity	$e_\oplus = 0.0167$

Moon

Radius	$R_☾ = 1737$ km
Orbital period	$T_☾ = 27.32^\mathrm{d}$
Orbital inclination	$i_☾ = 5.1^\circ$

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Absolute magnitude	$M_\odot = 4.74^\mathrm{m}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Emission constants

Stefan–Boltzmann	$\sigma = 5.67 \cdot 10^{-8}$ (W/m^2)/K^4
Wien's displacement	$b = 2898$ µm · K

UBV system

	Mean wavelengths
U band	$\lambda_U = 364$ nm
B band	$\lambda_B = 442$ nm
V band	$\lambda_V = 540$ nm

Hydrogen spectrum

Lyman Lα	$\lambda_{L\alpha} = 1215.7$ Å
Balmer Hα	$\lambda_{H\alpha} = 6562.8$ Å