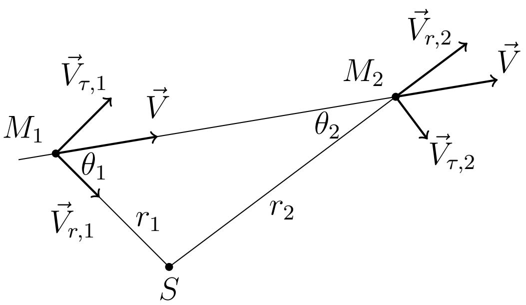
1 Slow down and shine

Proper motion of some star has reduced by 50 % in 10 000 years. How much has the apparent magnitude of the star changed? Assume that the star was moving linearly at a constant speed.

Solution:



Proper motion μ is related to the tangential velocity V_τ of the star and the distance r to it:

$$\mu \propto \frac{V_\tau}{r}.$$

Let M_1 be the initial position of the star, M_2 be the position of the star after 10 000 years. The tangential velocity $V_{\tau,i}$ is related to the total velocity V:

$$V_{\tau,1} = V \sin\theta_1 , \qquad V_{\tau,2} = V \sin\theta_2 .$$

The ratio of the proper motions is equal to 0.5,

$$0.5 = \frac{\mu_2}{\mu_1} = \frac{V_{\tau,2}}{V_{\tau,1}} \cdot \frac{r_1}{r_2} = \frac{\sin\theta_2}{\sin\theta_1} \cdot \frac{r_1}{r_2} .$$

Using the law of sines, we express the ratio of the sine angles in terms of the ratio of the sides of the $\triangle M_1 M_2 S$:

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{r_1}{r_2} .$$

Therefore,

$$0.5 = \frac{r_1}{r_2} \cdot \frac{r_1}{r_2} = \left(\frac{r_1}{r_2}\right)^2 .$$

So we conclude that $r_2 = \sqrt{2} r_1$. The illuminance is inversely proportional to the square of the distance, so the illuminance also halved. The change in magnitude is

$$\Delta m = 2.5 \lg 2 = 0.75^{\mathrm{m}}.$$

Marking Scheme:

- Relation between proper motion and distance — **3 pt.**
- Relation between angles and distances — **3 pt.**
- Correct use of Pogson's equation — **3 pt.**
- Correct answer — **1 pt.**

2 Mysterious Mercury

Here are a few celestial events for October 2022:

October 9 — Mercury at greatest elongation;
October 24 — Mercury occultation by the Moon;
October 25 — partial solar eclipse.

When was the last greatest evening elongation of Mercury before the olympiad? In what constellation was Mercury at that time? Assume Mercury's orbit is circular with radius $r_M = 0.387$ au.

Solution:

The synodic period of Mercury is

$$S = \frac{1}{\frac{1}{T_M} - \frac{1}{T_\oplus}}.$$

Using Kepler's third law, we obtain

$$S = \frac{T_\oplus}{\left(\frac{a_M}{a_\oplus}\right)^{-3/2} - 1} = \frac{365.26^\mathrm{d}}{0.387^{-1.5} - 1} = 115.82^\mathrm{d} \approx 116^\mathrm{d}.$$

The occultation of Mercury by the Moon will occur the day before the eclipse. Consequently, Mercury at this time is west of the Sun and is observed in the morning.

Let us calculate the time interval between the eastern and western elongations of Mercury. Mercury has to orbit the angle $2\arccos\frac{a_M}{a_\oplus} = 2\arccos 0.387 = 134^\circ$ relative to the Earth.

It will pass this angle in $\frac{134^\circ}{360^\circ} \cdot S = 43^\mathrm{d}$, as S is period of relative motion of Mercury and Earth, and Mercury orbits around the Sun faster than the Earth.

Thus, the last greatest eastern (evening) elongation of Mercury occurred 43 days before October 9, on August 27, 2022. We are lucky, this is an absolutely accurate answer, despite the actual ellipticity of Mercury's orbit!

At that time Mercury was $\arcsin 0.387 \approx 23^\circ$ east of the Sun, so it has the ecliptic longitude of $\approx 160^\circ + 23^\circ = 183^\circ$. It corresponds to Virgo.

Marking Scheme:

- Concept of synodic period — **1 pt.**
- Calculation of synodic period — **2 pt.**
- Description or geometric scheme of the situation — **2 pt.**
- The moment of the greatest evening elongation — **2 pt.**
- Identification of the constellation — **3 pt.**

3 Anti-Earth

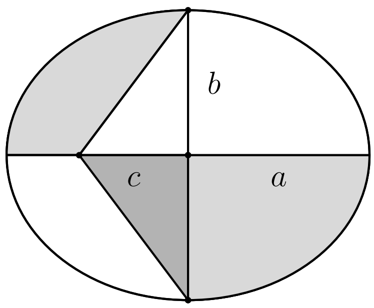
Imagine that Anti-Earth really exists and moves exactly along the orbit of the Earth so that it passes aphelion at the moment when the Earth passes perihelion and vice versa. Neglect the gravitational interaction between the Earth and Anti-Earth.

Let the Earth be at perihelion, Anti-Earth at aphelion at some point in time. Which of them and how much earlier will cross the minor axis of the orbit? Can observers detect Anti-Earth when the Earth crosses the minor axis of the orbit?

Solution:

Using Kepler's second law and observing the symmetry, one may conclude that

$$\tau_E = \left(\frac{1}{4} - \frac{e}{2\pi}\right) T_\oplus,$$
$$\tau_A = \left(\frac{1}{4} + \frac{e}{2\pi}\right) T_\oplus,$$



as an area of a quarter of the ellipse is $A_1 = \frac{A}{4} = \frac{\pi}{4}ab$, and the area of the highlighted triangle is $A_2 = \frac{1}{2}bc = \frac{e}{2}ab = \frac{e}{2\pi}A$.

The Earth crosses the minor axis earlier,

$$\Delta\tau = \frac{e}{\pi} T_\oplus = 365.26^{\mathrm{d}} \cdot \frac{0.0167}{3.1416} = 1.94^{\mathrm{d}} \approx 2^{\mathrm{d}}.$$

The distance between the center of the Sun and the center of the Earth's orbit is

$$c = ea = 0.0167 \cdot 149.6 \cdot 10^6 \text{ km} = 2.5 \cdot 10^6 \text{ km} = 3.6 R_\odot.$$

Therefore, Anti-Earth is definitely visible next to the Sun at that moment. The fact that it did not reach the minor axis only improves the conditions for its visibility.

Marking Scheme:

- Correct understanding of the situation, providing the correct diagram — **2 pt.**
- Estimation of the time interval.

  In case of using Kepler's second law:

  - The law is properly mentioned or formulated — **1 pt,**
  - Difference of areas and time intervals — **2 pt,**
  - Correct value of the time interval — **1 pt.**

In case of using Kepler's equation:

- The equation is correct — **1 pt,**
- Difference of mean anomalies is calculated — **2 pt,**
- Correct value of the time interval — **1 pt.**

• Describing the visibility:

- Full justification of visibility is given — **3 pt:**
  The linear eccentricity is calculated and compared with the position of the Sun,
  OR the eccentric anomaly is calculated and it is shown that it exceeds 90° by more than the angular radius of the Sun,
  OR the true anomaly is calculated from the eccentric and the cosine theorem is used.
  Particular case: just the fact that the Earth moves 2° in 2 days — 1 pt.
- Correct answer — **1 pt.**

4 Red Skies

A close binary star consists of two components with the same brightness in the V band:

№	Spectral class	True $B-V$	True $U-B$
1	M2	$+1.8$	$+2.1$
2	B8	-0.1	-0.6

Determine the color indices of the binary star after passing through the Earth's atmosphere at an altitude of 45°. Which of the components has a greater color excess due to atmospheric absorption? Consider only Rayleigh scattering $\sigma \propto \lambda^{-4}$, with absorption $A_V = 0.30^{\mathrm{m}}$ in the V band at the zenith.

Solution:

Let V be the true (extra-atmospheric) magnitude of each component in the V band. Then we obtain true magnitudes of each component in the U and B bands.

№	True U	True B	True V
1	$V+3.9$	$V+1.8$	V
2	$V-0.7$	$V-0.1$	V

Total magnitude m_t can be calculated from the magnitudes of components m_1 and m_2 as follows (each light source is compared here to 0^{m}):

$$m_t = -2.5 \log \left(10^{-0.4 m_1} + 10^{-0.4 m_2}\right).$$

Then consider $m_1 = V + a,\ \ m_2 = V + b$:

$$m_t = V - 2.5 \log \left(10^{-0.4 a} + 10^{-0.4 b}\right).$$

After the calculation, it is possible to determine the true color indices of the binary star:

№	True U	True B	True V	True $B-V$	True $U-B$
Total	$V-0.72$	$V-0.27$	$V-0.75$	0.48	-0.45

Please note that "total" color indices lay between the corresponding color indices of components.

Optical depth $\tau \propto \sigma L$, where σ is scattering cross-section and L is beam path length. Recalling Beer's law,

$$I = I_0 \cdot e^{-\tau}, \tag{1}$$

and in view of the logarithmic nature of stellar magnitudes, we conclude that the change in magnitude due to atmospheric absorption

$$\Delta m \propto \tau \propto \sigma L \propto \lambda^{-4} L.$$

At an altitude of 45°, we estimate the path of the beam in the atmosphere as

$$L_{45} \simeq \frac{L_{90}}{\sin 45^\circ} = \sqrt{2} L_{90}.$$

Let us calculate Δm for different bands, taking the mean band wavelength as an estimate:

$$\Delta m_{b,45} = \sqrt{2} A_V \cdot \left(\frac{\lambda_V}{\lambda_b}\right)^4.$$

Band	U	B	V	$\Delta(U-B)$	$\Delta(B-V)$
Δm	2.05	0.95	0.42	1.10	0.53

Therefore, after passing through the atmosphere, the color indices of the binary star are

$$(B-V)' = (B-V) + \Delta(B-V) = 0.48 + 0.53 = 1.0;$$
$$(U-B)' = (U-B) + \Delta(U-B) = -0.45 + 1.10 = 0.65.$$

Color excess is a characteristic of the absorbing medium, not of a particular radiation source, so the color excesses of the components match.

Marking Scheme:

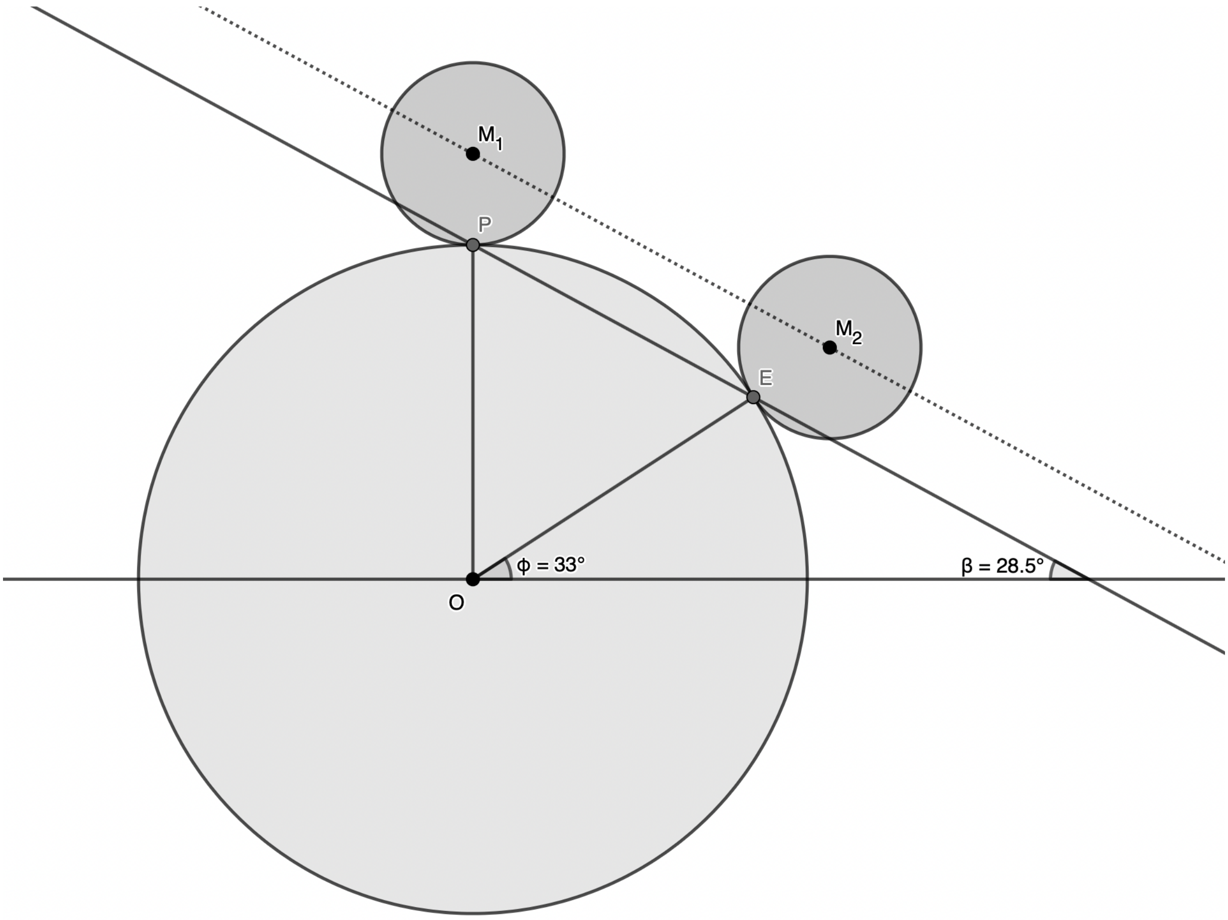
- True colors of the binary:
  - Correct formula for the color — **2 pt,**
  - Correct values of the colors — **0.5 pt + 0.5 pt.**
- $\approx \sqrt{2}$-fold increase of the absorption at an altitude of 45° — **1 pt.**
- Relation between absorption and wavelength — **2 pt.**
- Proof of equality of color excesses:
  - Full justification — **1 pt,**
  - Correct answer — **1 pt.**
- Color excesses estimation — **0.5 pt + 0.5 pt.**
- Colors after passing through the atmosphere — **0.5 pt + 0.5 pt.**

5 Follow the Shadow

Occultation of a star by the Moon begins on Earth at the North Pole, and ends at a point with latitude 33° North on the prime meridian (0° longitude) at 0^{h} local time. Find the star's equatorial coordinates and the occultation date.

Solution:

Let us look at the occultation of the star by the Moon from the star itself:



The Moon moves at an angle

$$\beta = \frac{90^\circ - \phi}{2} = 28.5^\circ = \varepsilon + i_{☾}.$$

to the celestial equator. Then it is clear that the Moon is near the descending node of its orbit, which coincided with the point of the autumnal equinox.

At the end of the occultation, the Sun is over the meridian opposite to the meridian of point E, because at the point E it is midnight at that moment. Therefore, the Sun is at the point of the summer solstice. The date is June 22.

Marking Scheme:

- The Moon is near the autumnal equinox (without justification, taking into account the inclination of the lunar orbit to the ecliptic) — **2 pt.**
- The Moon is near a descending node of the orbit (as justification of the previous statement) — **2 pt.**
- The Moon is in the first quarter (or equivalent statement) — **2 pt.**
- The coordinates of the star — **2 pt.**
- The occultation date — **2 pt.**

6 Orbital Time Machine

Light-collecting area of the James Webb Space Telescope (JWST) is about 25 m^2. Estimate how long it takes to catch 10 photons emitted by a solar type star in a galaxy at redshift $z = 0.2$.

The JWST wavelength coverage is 0.6 – 28.5 μm (orange to mid-infrared). Assume that the JWST is equipped with a single detector capable of registering photons over the entire wavelength coverage of the telescope.

Solution:

First, let us estimate the distance to the star. For simplicity, we will assume here and below that $z \ll 1$ still:

$$d = \frac{cz}{H_0} = \frac{3 \cdot 10^5 \text{ km/s} \cdot 0.2}{70 \text{ (km/s)/Mpc}} = 0.86 \text{ Gpc}.$$

We consider the star to be a black body. Average energy of a black-body photon is $\bar{\varepsilon} = 2.70 k_B T$. The estimated photon flux density (number of photons per square meter per second) from the star in the Solar system is

$$F = \frac{L_\odot}{4\pi d^2} \cdot \frac{1}{\bar{\varepsilon}} = \frac{4\pi R_\odot^2 \sigma T_\odot^4}{4\pi d^2 \cdot 2.70 k_B T_\odot} = \frac{R_\odot^2 \sigma T_\odot^3}{2.70 d^2 k_B} = 2.0 \cdot 10^{-7} \text{ m}^{-2} \cdot \text{s}^{-1}.$$

Not all of these photons fall within the telescope's wavelength coverage. The observed λ range of 0.6 – 28.5 μm corresponds to initial $\lambda' = \frac{\lambda}{1+z}$ range of 0.5 – 23.75 μm.

Recalling Planck's law and expression $\varepsilon = h\nu = \dfrac{hc}{\lambda}$ for energy of a photon, we introduce substitution $x \equiv \frac{hc}{\lambda k_B T_\odot}$:

$$x_1 = 0.105, \quad x_2 = 4.97.$$

Such a range covers both the mean value over the energy distribution ($x_{\text{mean}} = 2.70$, mentioned before) and the rather sharp peak of the distribution itself ($x_{\max} = 2.82$). So, we can expect that in fact almost all photons fall within the JWST wavelength coverage (in fact, 90 % of the photons from the star meet that condition).

By the way, the frequency for maximal spectral radiance of a black body with temperature $T_\odot$ is $\nu_{\max} = 341$ THz with corresponding wavelength $\lambda'_{\max} = 0.88$ μm. The redshift only shifts this peak further into the infrared.

Note. Constants k_B and σ are not given in the table. They can be estimated using well-known energetic properties of the Sun. For example, the solar luminosity

$$L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4 \approx 3.8 \cdot 10^{26} \text{ W},$$

or the solar constant

$$A_\odot \approx 1.4 \text{ kW/m}^2 = \frac{R_\odot^2 \sigma T_\odot^4}{a_\oplus^2}.$$

So, the Stefan–Boltzmann constant

$$\sigma = \frac{L_\odot}{4\pi R_\odot^2 T_\odot^4} = \frac{A_\odot a_\oplus^2}{R_\odot^2 T_\odot^4} \approx 5.7 \cdot 10^{-8}\ \mathrm{W/(m^2 \cdot K^4)}.$$

One could also recall Wien's displacement law:

$$\frac{hc}{\lambda_{\max}} \simeq 5.0 k_B T_\odot,$$

where $\lambda_{\max} = 0.5$ μm for the solar spectrum. Of course, $\lambda_{\max} \neq \lambda'_{\max}$: it is well-known that parameterization by frequency implies a different maximal wavelength than parameterization by wavelength.

Then $x_i = \frac{\lambda_{\max}}{\lambda_i} \cdot 5.0$, and

$$k_B = \frac{hc}{5\lambda_{\max} T_\odot} \approx 1.4 \cdot 10^{-23}\ \mathrm{J/K}.$$

Actually, the exact expression for F in this model is

$$F = \frac{R_\odot^2 \sigma T_\odot^4}{d^2} \cdot \left(\frac{\int_0^\infty \frac{x^3\,dx}{e^x-1}}{\int_0^\infty \frac{x^2\,dx}{e^x-1}} k_B T_\odot \right)^{-1} \frac{\int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1}}{\int_0^\infty \frac{x^2\,dx}{e^x-1}} = \frac{R_\odot^2 T_\odot^3}{d^2 k_B} \cdot \frac{2\pi k_B^4}{h^3 c^2} \cdot 3! \cdot \zeta(4) \cdot \frac{2! \cdot \zeta(3)}{3! \cdot \zeta(4)} \cdot \frac{\int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1}}{2! \cdot \zeta(3)} =$$

$$= \frac{2\pi R_\odot^2 c}{d^2} \cdot \left(\frac{k_B T_\odot}{hc}\right)^3 \cdot \int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1} = \frac{2\pi R_\odot^2 c}{125 d^2 \lambda_{\max}^3} \cdot \int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1} \simeq 0.83 \cdot 10^{-7}\ \mathrm{m^{-2} \cdot s^{-1}} \times \int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1}.$$

The last integral can be calculated numerically, although this was not required for the presented solution. One could also estimate that integral as

$$\int_{x_1}^{x_2} \frac{x^2\,dx}{e^x-1} \approx \int_0^\infty \frac{x^2\,dx}{e^x-1} \approx \int_0^\infty x^2 e^{-x}\,dx = 2! = 2 \sim 1.$$

How long does it take for the telescope ($A = 25$ m^2) to accumulate 10 photons from the star?

$$\tau = \frac{10}{FA} = 2.0 \cdot 10^6\ \mathrm{s} \approx 23\ \mathrm{d}.$$

Marking Scheme:

- Distance to the galaxy — **2 pt.**
- Illuminance — **3 pt.**
- Photon flux (taking into account the wavelength coverage and redshift) — **3 pt.**
- Time interval — **2 pt.**

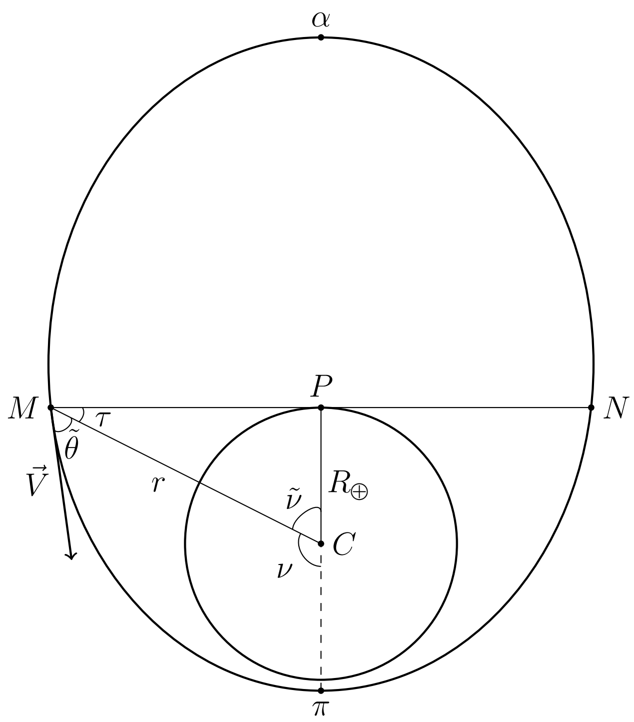
7 Polar Wi-Fi

A satellite is moving in a polar geocentric orbit with semi-major axis $a = 15\,400$ km, eccentricity $e = 0.55$, and argument of pericenter $\omega = 270°$. Imagine that an observer is at the North Pole of the Earth. The satellite emits a signal with a frequency of 2.4 GHz. At some point in time, the satellite is observed on the horizon.

Estimate

a) the distance from the observer to the satellite,
b) the total velocity of the satellite at that moment,
c) the shift in the observed signal frequency.

Solution:



a) The observer is at point P. We should determine the geocentric distance $r \equiv MC$.

$$\begin{cases} R_\oplus = r\cos\tilde{\nu} = -r\cos\nu, \\ r = \dfrac{a(1-e^2)}{1+e\cos\nu} \end{cases} \Rightarrow \quad r = \frac{a(1-e^2)r}{r-eR_\oplus} \quad \Rightarrow \quad r = a(1-e^2) + eR_\oplus;$$

$$r = 15\,400 \text{ km} \cdot (1 - 0.55^2) + 6\,371 \text{ km} \cdot 0.55 = 14\,246 \text{ km}.$$

Therefore, the distance from the observer to the satellite is

$$MP = \sqrt{r^2 - R_\oplus^2} = \sqrt{(14\,246 \text{ km})^2 - (6\,371 \text{ km})^2} = 12\,742 \text{ km}.$$

b) Let us write the vis-viva equation. Here V is the total velocity of the satellite:

$$V^2 = GM_\oplus \cdot \left(\frac{2}{r} - \frac{1}{a}\right) = gR_\oplus^2 \cdot \left(\frac{2}{r} - \frac{1}{a}\right); \tag{2}$$

$$V = \sqrt{9.81 \text{ m/s} \cdot 6\,371 \text{ km} \cdot \left(\frac{2}{1.4246 \cdot 10^7 \text{ m}} - \frac{1}{1.5400 \cdot 10^7 \text{ m}}\right)} = 5.48 \text{ km/s}.$$

c) First, we should determine the line-of-sight velocity of the satellite. Let us find the angle between the direction of the velocity vector of the satellite and the observer's line of sight.

Let θ be the angle between the velocity vector and the radius vector. In the figure, the symbol $\tilde{\theta} \equiv 180^\circ - \theta$. We may write the conservation of angular momentum law for the current point and the pericenter:

$$V^2 r^2 \sin^2\theta = V_\pi^2 r_\pi^2;$$

$$V^2 r^2 \sin^2\theta = \frac{GM_\oplus}{a} \cdot \frac{1+e}{1-e} \cdot a^2(1-e)^2 = GM_\oplus a(1-e^2). \tag{3}$$

Next, we substitute the expression for the velocity (2) into the formula (3) above:

$$gR_\oplus^2 \cdot \left(\frac{2}{r} - \frac{1}{a}\right) \cdot r^2 \sin^2\theta = gR_\oplus^2 \cdot a(1-e^2) \quad \Rightarrow \quad \sin^2\theta = \frac{a(1-e^2)}{r^2\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

In our case

$$\sin^2\theta = \frac{1.5400 \cdot 10^7 \text{ m} \cdot (1 - 0.55^2)}{(1.4246 \cdot 10^7 \text{ m})^2 \cdot \left(\frac{2}{1.4246 \cdot 10^7 \text{ m}} - \frac{1}{1.5400 \cdot 10^7 \text{ m}}\right)} = 0.701 \quad \Rightarrow \quad \theta = 123.1^\circ.$$

Next, we estimate the angle τ between the radius vector of the satellite and the line of sight.

$$\begin{cases} \tau = 90^\circ - \tilde{\nu}, \\ \tilde{\nu} = \arccos\dfrac{R_\oplus}{r} = \arccos\dfrac{6\,371 \text{ km}}{14\,246 \text{ km}} = 63.4^\circ \end{cases} \quad \Rightarrow \quad \tau = 90^\circ - 63.4^\circ = 26.6^\circ.$$

The angle between the direction of the velocity vector of the satellite and the observer's line of sight ψ is $\theta + \tau = (180^\circ - 123.1^\circ) + 26.6^\circ = 83.5^\circ$, or $180^\circ - 83.5^\circ = 96.5^\circ$ in the symmetrical case. Finally, the radial velocity of the satellite

$$V_r = V\cos\psi = 5.48 \text{ km/s} \cdot \pm\cos 83.5^\circ = \pm 0.62 \text{ km/s}.$$

The shift in the observed frequency due to radial motion of the satellite is

$$\Delta f = \frac{V_r}{c} f = 2.4 \cdot 10^9 \text{ Hz} \cdot \frac{\pm 0.62 \text{ km/s}}{3 \cdot 10^5 \text{ km/s}} \approx \pm 5 \text{ kHz}.$$

Marking Scheme:

- Correct orbital diagram is given (aphelion at zenith) — **2 pt.**
- Distance:
  - True anomaly — **1 pt,**
  - Geocentric distance — **1 pt,**
  - Distance from the observer to the satellite — **1 pt.**
- Total velocity:
  - Correct formulae (vis-viva equation or similar) — **1 pt,**
  - Correct answer — **1 pt.**
- Shift in observed frequency:
  - Correct formula and value of radial velocity — **1 pt,**
  - Doppler shift — **1 pt,**
  - Two possible cases of observation (rise/set) — **1 pt.**

8 Perfect Nebula

A planet with synchronous rotation orbits a star in a circular orbit. The planet has no atmosphere, but the system is surrounded by a large homogeneous spherical gaseous nebula. This nebula does not absorb light, but only scatters it isotropically and with the same scattering properties for all wavelengths.

The surface temperature of the planet at the point facing the star is twice that of the opposite point. The Sun's apparent magnitude on the planet's surface is 20^{m}. Find the distance to the system. Ignore interstellar extinction.

Solution:

Let us assume that the radius of nebula l is much more than radius of planet's orbit R. The star luminosity is B, optical depth of nebula is τ. The planet is inside the nebula, so the energy flux from the star near the planet is

$$J_S = \frac{B}{4\pi R^2}.$$

The energy scattered by the nebula is

$$B_N = B\left(1 - e^{-\tau}\right).$$

Angular element $d\Omega$ of the nebula scatters

$$dB_N = \frac{d\Omega}{4\pi} B_N,$$

and that resuls in illuminance near the planet

$$dJ_N = \frac{dB_N}{4\pi l^2}.$$

Energy flux on the planet surface from an angular element is

$$dJ_N \cdot \cos\theta = B\left(1 - e^{-\tau}\right) \cdot \cos\theta \cdot \pi^2 l^2 \cdot d\Omega.$$

Thus, total amount from sky hemisphere is

$$J_N = \int\limits_{\text{Hemisph.}} dJ_N \cdot \cos\theta = \frac{1 - e^{-\tau}}{32\pi^2 l^2} B.$$

Here the fact that average $\langle \cos\theta \rangle = \frac{1}{2}$ over the hemisphere is used.

The subsolar point of the planet receives $J_S + J_N$, while the opposite point receives J_N. Recalling Stefan–Boltzmann's law, we conclude that

$$J_S + J_N = 2^4 \cdot J_N \quad \Rightarrow \quad J_S = 15 J_N.$$

Therefore,

$$1 - e^{-\tau} = \frac{8\pi l^2}{15R^2}.$$

The latter equation has no solutions for τ, unless we consider the "close model" $l \gtrsim R$ with rough approximation. In this case the nebula fragment near the planet can be considered as a flat layer with optical depth $\tau < 1$, illuminated with the energy flux J_S. It scatters τ-part of the incident flux to the outer and inner side of the nebula:

$$J_{N0} = \frac{\tau}{2} J_S.$$

To meet the temperature relation,

$$J_S = 16 J_{N0},$$

this gives the lower limit value of $\tau_{\min} = \frac{1}{8}$. Generally speaking, we have to write the final condition on the optical depth:

$$\tau > \frac{1}{8}.$$

The Sun has the absolute magnitude $M_\odot = +4.7^\mathrm{m}$ and apparent magnitude $m = +20^\mathrm{m}$. Taking the logarithm of Beer's law (1), we obtain the equation for the distance r:

$$m = M - 5 + 5 \lg r + 2.5 \lg e \cdot \tau.$$

Finally,

$$\lg r < 4.03 \quad \Rightarrow \quad r \lesssim 11 \text{ kpc}.$$

Marking Scheme:

- Distance determination, without the absorption — **2 pt.**
- Estimation of the ratio of energy flows — **2 pt.**
- Understanding that the condition can be fulfilled at different thicknesses, there is only a lower limit — **2 pt.**
- Adequate energy transfer model (not necessarily the same as above) — **2 pt.**
- Distance determination, taking absorption into account:
  - Inequality — **2 pt,**
  - Equality only — 1 pt. *in particular.*

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Hubble constant	$H_0 = 70$ (km/s)/Mpc
Astronomical unit	1 au = $149.6 \cdot 10^6$ km
Parsec	1 pc = 206 265 au

Earth

Radius	$R_\oplus = 6371$ km
Obliquity	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^\mathrm{d}$
Orbital eccentricity	$e_\oplus = 0.0167$

Moon

Radius	$R_☾ = 1737$ km
Orbital period	$T_☾ = 27.32^\mathrm{d}$
Orbital inclination	$i_☾ = 5.1^\circ$

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Absolute magnitude	$M_\odot = 4.74^\mathrm{m}$
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K

UBV system	Mean wavelengths
U band	$\lambda_U = 364$ nm
B band	$\lambda_B = 442$ nm
V band	$\lambda_V = 540$ nm