1 Ceres

A spacecraft is in a circular orbit with radius $r = 1.3$ au in the ecliptic plane. An additional speed of 5 km/s is instantly given to the spacecraft in the direction of movement, so its orbit intersects with the orbit of 1 Ceres. What will be the relative velocity of the spacecraft and Ceres when the spacecraft crosses its orbit for the first time? The distance between Ceres and the spacecraft at that time is 10^6 km. Assume the orbit of Ceres to be circular with radius $r_C = 2.8$ au. The directions of orbital motion of the objects are the same.

Solution:

First, we determine the parameters of the orbit along which the spacecraft will move after increasing its velocity. Let us write the law of conservation of energy per unit mass of the spacecraft:

$$\frac{V^2}{2} - \frac{G\mathfrak{M}_\odot}{r} = -\frac{G\mathfrak{M}_\odot}{2a} \tag{1}$$

where r is the current distance from the Sun and a is the semi-major axis of the orbit.

Note. The mass of the Sun is not indicated in the table. Although its value is well known, it can be easily estimated using the Earth's orbital parameters. By Kepler's third law,

$$T_\oplus = 2\pi\sqrt{\frac{(1\text{ au})^3}{G\mathfrak{M}_\odot}} \quad \Rightarrow \quad \mathfrak{M}_\odot = \left(\frac{2\pi}{T_\oplus}\right)^2 \cdot \frac{(1\text{ au})^3}{G} = 2.0 \cdot 10^{30}\text{ kg}.$$

We should estimate the velocity of the spacecraft. The velocity in the initial circular orbit

$$V_0 = \sqrt{\frac{G\mathfrak{M}_\odot}{r}} = 2.6 \cdot 10^4\text{ m/s}.$$

The total velocity after the correction is

$$V_\pi = V_0 + 5.0 \cdot 10^3\text{ m/s} = 3.1 \cdot 10^4\text{ m/s}.$$

Therefore, taking into account (1) the semi-major axis of the elliptic orbit is

$$a = \frac{G\mathfrak{M}_\odot r}{2G\mathfrak{M}_\odot - V_\pi^2 r} = 3.4 \cdot 10^{11}\text{ m} = 2.24\text{ au}.$$

At the initial moment of time, the velocity vector was perpendicular to the heliocentric radius vector. The final velocity is also perpendicular to the radius vector, that is, after increasing the velocity, the spacecraft starts moving from the perihelion of the orbit, $r_\pi := r$.

Next, we estimate the eccentricity of the orbit:

$$r_\pi = a(1-e) \quad \Rightarrow \quad e = 1 - \frac{r_\pi}{a} = 1 - \frac{1.3}{2.24} = 0.42.$$

Let us determine the velocity V_1 of the spacecraft at the moment of crossing the orbit of Ceres. According to the vis-viva equation,

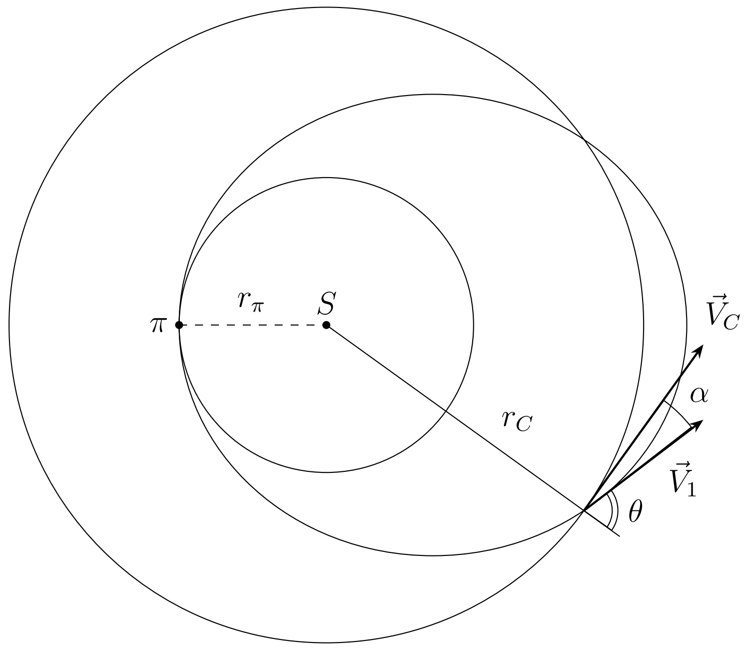
$$V_1^2 = G\mathfrak{M}_\odot \left(\frac{2}{r_C} - \frac{1}{a}\right);$$

$$V_1 = \sqrt{6.67 \cdot 10^{-11} \times 2.0 \cdot 10^{30} \times \left(\frac{2}{2.8 \cdot 1.496 \cdot 10^{11}} - \frac{1}{2.24 \cdot 1.496 \cdot 10^{11}}\right)} = 1.5 \cdot 10^4 \text{ m/s}.$$

Next, we determine the angle θ between the velocity vector and the radius vector:

$$V_1 r_C \sin\theta = V_\pi r_\pi = \sqrt{G\mathfrak{M}_\odot a(1-e^2)} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{G\mathfrak{M}_\odot a(1-e^2)}}{V_1 r_C};$$

$$\theta = \arcsin \frac{\sqrt{6.67 \cdot 10^{-11} \times 2.0 \cdot 10^{30} \times 2.24 \cdot 1.496 \cdot 10^{11} \times (1 - 0.42^2)}}{1.54 \cdot 10^4 \times 2.8 \cdot 1.496 \cdot 10^{11}} = \arcsin 0.939 = 70^\circ.$$



On the scale of the orbit, the distance of 10^6 km is very small, so we can assume that Ceres and the spacecraft are almost at the same point of the orbit. At the same time, this distance is large enough that we can ignore the attraction of Ceres. Indeed, the radius of the Earth's sphere of influence is $\approx 10^6$ km. Ceres is only 3 *times* farther from the Sun than the Earth, but its mass is 4 *orders of magnitude* less than the Earth's[1].

The velocity of Ceres

$$V_c = \sqrt{\frac{G\mathfrak{M}_\odot}{r_c}} = 1.8 \cdot 10^4 \text{ m/s},$$

[1] The conclusion is quite obvious, so there is no need to calculate the exact radius of the sphere of influence.

is perpendicular to the radius vector. Then the angle between the velocity vectors of the spacecraft and Ceres at the approach is equal to $\alpha = 180^\circ - \theta - 90^\circ = 20^\circ$.

Therefore, the relative velocity will be

$$\Delta V = \sqrt{V_c^2 + V_1^2 - 2V_c V_1 \cos\alpha} =$$
$$= \sqrt{(1.8 \cdot 10^4)^2 + (1.5 \cdot 10^4)^2 - 2 \times (1.8 \cdot 10^4) \times (1.5 \cdot 10^4) \times \cos 20^\circ} = 6.5 \cdot 10^3 \text{ m/s}.$$

Marking Scheme:

- Velocity of Ceres V_c — **1 pt.**
- Velocity of the spacecraft in the initial circular orbit V_0 — **1 pt.**
- Velocity of the spacecraft after the correction V_π — **1 pt.**
- Semi-major axis of the elliptic orbit a or proper use of energy convervation — **1 pt.**
- Eccentricity e or proper use of angular momentum convervation — **1 pt.**
- Velocity of the spacecraft crossing the orbit of Ceres V_1 — **1 pt.**
- Angle between the velocity and the radius vectors θ — **1 pt.**
- Angle between the velocity vectors of the spacecraft and Ceres α — **1 pt.**
- Expression for ΔV and evaluation — **1 pt + 1 pt.**

2 Straight Forward

Currently, a star has a proper motion of $0.5''$/year with a parallax of $0.08''$. The hydrogen Hα line in the stellar spectrum is observed at wavelength $\lambda = 6561.0$ Å. Assume the star's velocity vector to be constant. Estimate the radial velocity of the star in 20 000 years.

Solution: First, we determine the current distance to the star from the known parallax ϖ:

$$r = \frac{1}{\varpi\,['']} = 12.5 \text{ pc}.$$

Then, let us find the current tangential velocity of the star. It is quite convenient to use the ratio 1 au = 1 pc × 1″:

$$V_{\tau,1} = \mu\,[''/\text{yr}] \cdot r\,[\text{pc}] = 0.5 \times \frac{1}{0.08}\,[\text{au/yr}] = 6.25\ [\text{au/yr}] = 29.6\ [\text{km/s}].$$ [2]

Next, we estimate the current radial velocity. According to the Doppler effect formula

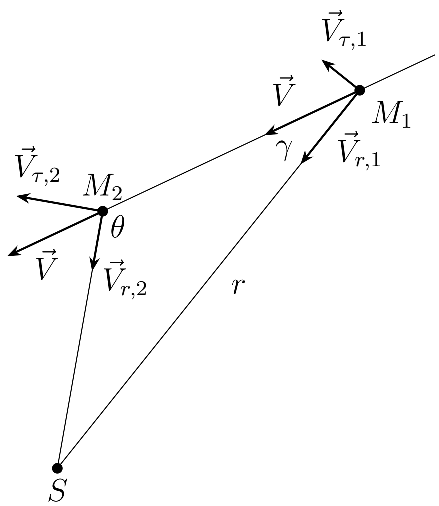
$$\frac{V_{r,1}}{c} = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} \quad \Rightarrow \quad V_{r,1} = c \cdot \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = 3 \cdot 10^5 \times \frac{6561.0 - 6562.8}{6562.8} = -82.3 \text{ km/s}.$$

The total velocity is

$$V = \sqrt{V_{\tau,1}^2 + V_{r,1}^2} = \sqrt{82.3^2 + 29.6^2} = 87 \text{ km/s}.$$

In 20 000 years, the star will cover a distance of

$$|M_1 M_2| = 87 \text{ km/s} \times 20\,000 \cdot (365.25 \times 24 \times 60 \times 60) \text{ s} = 5.5 \cdot 10^{13} \text{ km} = 1.8 \text{ pc}.$$



[2]It is also remarkable that 30 km/s $\approx V_\oplus = 2\pi$ au/yr ≈ 6.28 au/yr $\approx V_{\tau,1}$.

S denotes the Sun, M_1 is the current position of the star and M_2 is the position of the star in future. Let us write the law of cosines for $\triangle M_1M_2S$:

$$|M_2S|^2 = |M_1M_2|^2 + |M_1S|^2 - 2 \cdot |M_1M_2| \cdot |M_1S| \cos\gamma$$

where $|M_1S| = \dfrac{1}{\varpi\,['']} = 12.5$ pc, $\gamma = \arccos\dfrac{V_{r,1}}{V} = 19^\circ$.

Consequently,

$$|M_2S|^2 = 1.8^2 + 12.5^2 - 2 \cdot 1.8 \cdot 12.5 \cos 19^\circ \Rightarrow |M_2S| = 10.8 \text{ pc}.$$

Next, we write the law of cosines for the same triangle:

$$|M_1S|^2 = |M_1M_2|^2 + |M_2S|^2 - 2 \cdot |M_1M_2| \cdot |M_2S| \cos\theta;$$

$$\cos\theta = \frac{|M_1M_2|^2 + |M_2S|^2 - |M_1S|^2}{2 \cdot |M_1M_2| \cdot |M_2S|} = \frac{1.8^2 + 10.8^2 - 12.5^2}{2 \cdot 1.8 \cdot 10.8} = -0.935 \quad \Rightarrow \quad \theta = 159^\circ.$$

The radial velocity is

$$V_{r,2} = V\cos\theta = 87 \cdot \cos 159^\circ = -81 \text{ km/s}.$$

Marking Scheme:

- Current distance to the star $|SM_1| \equiv r$ — **1 pt.**
- Current tangential velocity of the star $V_{\tau,1}$ — **1 pt.**
- Current radial velocity of the star $V_{r,1}$ — **1 pt.**
- Geometry of the problem (preferably, a drawing) — **1 pt.**
- Distance covered in 20 000 years $|M_1M_2|$ — **1 pt.**
- Direction of the star's velocity vector, angle γ — **1 pt.**
- Distance to the star in 20 000 years $|SM_2|$ — **2 pt.**
- Angle θ — **1 pt.**
- Radial velocity of the star in 20 000 years $V_{r,2}$ — **1 pt.**

3 Apparently Invisible

A planet orbits a main sequence star with apparent bolometric magnitude $m = +7^{\mathrm{m}}$ in a circular orbit with period $T = 570$ years. The star is 385 light years from the Sun.

a) Estimate the maximum angular distance between the planet and the star for an observer on the Earth.

b) Compare the obtained value with the angular resolution of the James Webb Space Telescope (JWST) at wavelength $\lambda = 3.8$ µm.

The effective diameter of JWST is $D = 6.5$ m. Neglect the interstellar extinction.

Solution:

a) In order to determine the maximum angular distance between a star and a planet, we need to know the radius of the planet's orbit, and for this we need to estimate the mass of the star. Recalling[3] that 1 pc = 3.26 light years, we estimate the luminosity based on the apparent magnitude and distance, namely

$$M = m + 5 - 5\lg r = 7 + 5 - 5\lg\frac{385}{3.26} = 1.6^{\mathrm{m}},$$

$$M - M_{\odot} = -2.5\lg\frac{L}{L_{\odot}} \quad\Rightarrow\quad L = 10^{0.4(M_{\odot}-M)}L_{\odot} = 10^{0.4(4.7-1.6)}L_{\odot} = 17L_{\odot}.$$

According to the mass–luminosity relation for main sequence stars, for such luminosities there is a power law relationship between mass $\mathfrak{M}$ and luminosity L:

$$L \propto \mathfrak{M}^4.$$

Therefore, the mass of the star is

$$\mathfrak{M} = \left(\frac{L}{L_{\odot}}\right)^{1/4}\mathfrak{M}_{\odot} \approx 2\mathfrak{M}_{\odot}\,.$$

Next, we estimate the radius of the orbit:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G\mathfrak{M}} \quad\Rightarrow\quad a = \sqrt[3]{\frac{G\mathfrak{M}T^2}{4\pi^2}};$$

$$a = \sqrt[3]{\frac{6.67\cdot10^{-11}\times(2\times2.0\cdot10^{30})\times(570\times365.25\cdot24\cdot60\cdot60)^2}{4\pi^2}} = 1.3\cdot10^{13}\ \mathrm{m} = 87\ \mathrm{au}.$$

The maximum angular distance is

$$\alpha = \frac{a}{r} = \frac{87\ \mathrm{au}}{\frac{385}{3.26}\cdot206265\ \mathrm{au}} \approx 4\cdot10^{-6}\ \mathrm{rad} \approx 0.7''.$$

[3] 1 pc = 206 265 au = $3.086\cdot10^{13}$ km = $3.086\cdot10^{16}$ m = $1.03\cdot10^{8}$ s × c = 3.26 × c × 365.25 d := 3.26 ly.

b) Next, we estimate the angular resolution of JWST:

$$\beta \sim \frac{\lambda}{D} = \frac{3.8 \cdot 10^{-6}}{6.5} \approx 6 \cdot 10^{-7}\ \text{rad} \approx 0.1''.$$

It can be noticed that the maximum angular distance between the planet and the star exceeds the angular resolution of the telescope.

The problem discusses the planetary system ***HIP 65426,*** *the planet was observed directly (Carter et al., 2023, arXiv:2208.14990).*

Marking Scheme:

- Absolute magnitude of the star M — **1.5 pt.**
- Luminosity of the star L — **1.5 pt.**
- Mass estimation: $L \propto \mathfrak{M}^{\alpha}$
  - $\alpha \in [3.5;\ 4.0]$ — **1.5 pt.**
  - $\alpha \in [3.0;\ 3.5)$ — 1 pt.
- Radius of the orbit a — **1.5 pt.**
- Maximum angular distance α — **1 pt.**
- Angular resolution estimation β — **2 pt.**
- Conclusion about the resolution — **1 pt.**

4 Hot Potato

A main sequence star has radius $R = 3.9R_{\odot}$, effective temperature $T = 9520$ K, and parallax $\varpi = 0.011''$. An asteroid orbits the star with an orbital period of 1 year. The asteroid rotates quite rapidly. The asteroid's surface reflects $A = 30\,\%$ of incident radiation.

Estimate:

a) the average density of the star,

b) the orbital radius of the asteroid,

c) the effective surface temperature of the asteroid.

d) Is it possible to observe this star from the Earth with the naked eye?
The bolometric correction at a given temperature is approximately -0.15.

Solution:

a) First, we determine the luminosity of the star. According to the Stefan–Boltzmann law,

$$L = 4\pi R^2 \sigma T^4.$$

We can compare the parameters of the star with the parameters of the Sun, and obtain the ratio

$$\frac{L}{L_{\odot}} = \left(\frac{R}{R_{\odot}}\right)^2 \left(\frac{T}{T_{\odot}}\right)^4 \quad \Rightarrow \quad L = 3.9^2 \cdot \left(\frac{9520}{5800}\right)^4 L_{\odot} = 1.1 \cdot 10^2 L_{\odot}\,.$$

According to the mass–luminosity relation for main sequence stars, there is a power law relationship between mass $\mathfrak{M}$ and luminosity L. For the relevant mass range that is

$$\frac{L}{L_{\odot}} = \begin{cases} \left(\mathfrak{M}/\mathfrak{M}_{\odot}\right)^4, & 0.4 < \mathfrak{M}/\mathfrak{M}_{\odot} < 2; \\ 1.5\left(\mathfrak{M}/\mathfrak{M}_{\odot}\right)^{3.5}, & 2 < \mathfrak{M}/\mathfrak{M}_{\odot} < 20. \end{cases}$$

We do not know in advance which of the dependencies is valid for a given star, but in fact, this does not greatly affect the estimation:

$$L \approx \begin{cases} 3.2\mathfrak{M}_{\odot}, & \text{for } L \propto \mathfrak{M}^4, \\ 3.4\mathfrak{M}_{\odot}, & \text{for } L \propto \mathfrak{M}^{3.5}. \end{cases}$$

The value belongs to the scope of the second expression. One way or another, any such estimate is accepted.

The average density of the star is

$$\langle\rho\rangle = \frac{\mathfrak{M}}{V} = \frac{\mathfrak{M}}{\frac{4}{3}\pi R^3} = \frac{3.4 \cdot 2.0 \cdot 10^{30}\ \text{kg}}{\frac{4}{3}\pi\left(3.9 \times 6.96 \cdot 10^8\ \text{m}\right)^3} = 8 \cdot 10^1\ \text{kg/m}^3.$$

b) According to Kepler's third law,

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G\mathfrak{M}} \quad \Rightarrow \quad a = \sqrt[3]{\frac{G\mathfrak{M}T^2}{4\pi^2}}$$

$$a = \sqrt[3]{\frac{6.67\cdot 10^{-11} \times (3.4\times 2.0\cdot 10^{30}) \times (365.25\cdot 24\cdot 60\cdot 60)^2}{4\pi^2}} = 2.3\cdot 10^{11}\ \mathrm{m} = 1.5\ \mathrm{au}.$$

c) Radiation flux of the star at the orbit of the asteroid is given by $E = \dfrac{L}{4\pi a^2}$.
The absorption rate of the asteroid with radius r depends on its cross section and albedo A:

$$\mathcal{E} = E\cdot(1-A)\cdot\pi r^2.$$

At equilibrium, the absorption rate is equal to the emission rate of the asteroid blackbody radiation. Therefore,

$$\frac{L}{4\pi a^2}\cdot(1-A)\cdot\pi r^2 = 4\pi r^2\sigma T_a^4 \quad \Rightarrow \quad T_a = \sqrt[4]{\frac{L(1-A)}{16\pi a^2\sigma}};$$

$$T_a = \sqrt[4]{\frac{(1.1\cdot 10^2)\times(3.828\cdot 10^{26})\times(1-0.30)}{16\pi\cdot(2.3\cdot 10^{11})^2\times 5.67\cdot 10^{-8}}} = 665\ \mathrm{K}.$$

d) First, we estimate the bolometric absolute magnitude of the star according to the Pogson formula:

$$M_{\mathrm{bol}} = M_\odot - 2.5\lg\frac{L}{L_\odot} = 4.74^{\mathrm{m}} - 2.5\lg(1.1\cdot 10^2) = -0.36^{\mathrm{m}},$$

$$M_V = M_{\mathrm{bol}} - \mathrm{BC} = -0.36^{\mathrm{m}} + 0.15^{\mathrm{m}} = -0.21^{\mathrm{m}}.$$

Next, we write the relation of absolute magnitude, apparent magnitude and distance. Worth noting, we should try to take the interstellar extinction into account, so we consider the term $A_V r$. The interstellar medium is irregularly distributed[4], however, we may assume the averaged value of absorption to be $A_V \sim 2^{\mathrm{m}}/\mathrm{kpc} = 0.002^{\mathrm{m}}/\mathrm{pc}$. Therefore,

$$m_V = M_V - 5 + 5\lg r + A_V r = M_V - 5 - 5\lg\varpi + 0.002\cdot\frac{1}{\varpi} =$$

$$= -0.21^m - 5 - 5\lg 0.011 + 0.002\times\frac{1}{0.011} = 4.8^{\mathrm{m}}.$$

Such a star is visible to the naked eye as a faint point source.

[4] We may assume the star to be located in the galactic disk. The reason is that the star is quite massive and the lifetime on the main sequence is inversely proportional to $\mathfrak{M}^3$, so the star is quite young and it did not have time to decline much from the galactic plane where it was formed.

Marking Scheme:

- Average density:
  - Stefan–Boltzmann law — **1 pt.**
  - Mass estimation — **1 pt.**
  - Average density of the star $\langle\rho\rangle$ — **1 pt.**
- Orbital radius:
  - Kepler's third law — **1 pt.**
  - Orbital radius of the asteroid a — **1 pt.**
- Asteroid temperature:
  - Thermal equilibrium equation for the asteroid — **1 pt.**
  - Surface temperature of the asteroid T_a — **1 pt.**
- Naked eye observation:
  - Pogson formula — **1 pt.**
  - Taking BC into account — **1 pt.**
  - Conclusion about the visibility of the star — **1 pt.**

  There is no penalty for not taking into account interstellar absorption.

5 Hide and Seek

At the 50th parallel (50° N), some star rises just before Antares ($\alpha_A = 16^{\text{h}}\,29^{\text{m}};\ \delta_A = -26^\circ\,26'$) disappears behind the horizon, and it sets just when Sirius ($\alpha_S = 6^{\text{h}}\,45^{\text{m}};\ \delta_S = -16^\circ\,43'$) appears. The star is brighter than $+1.5^{\text{m}}$. Estimate the equatorial coordinates of the star. What is the name of this star? Consider the Earth to be perfectly spherical with no atmosphere.

Solution:

First we determine hour angle t_A of Antares setting. Using spherical law of cosines for the spherical triangle with vertices in the north celestial pole, zenith and Antares — navigational triangle, we have **[2 pt]**

$$\cos z_A = \sin\varphi \sin\delta_A + \cos\varphi\cos\delta_A\cos t_A. \tag{2}$$

We assume $z_A = 90^\circ$ because we neglect the refraction. Therefore, **[1 pt]**

$$\cos t_A = -\tan\varphi\tan\delta_A = 0.592 \quad\Rightarrow\quad t_A = 53.7^\circ.$$

Antares setting sidereal time is **[1 pt]**

$$s_1 = t_A + \alpha_A = \frac{53.7^\circ}{15^\circ/\text{h}} + 16^{\text{h}}\,29^{\text{m}} = 20.06^{\text{h}}.$$

Similarly, we estimate hour angle t_S and sidereal time s_2 at the time of Sirius rising above the horizon **[1 pt]:**

$$\cos t_S = -\tan\varphi\tan\delta_S = 0.359 \quad\Rightarrow\quad t_S = -69.0^\circ,$$
$$s_2 = \frac{-69^\circ}{15^\circ/\text{h}} + 6^{\text{h}}\,45^{\text{m}} = 2.15^{\text{h}}.$$

The sidereal time of the star's upper culmination is equal to its right ascension **[2 pt]:**

$$\alpha_\star = s_c = \frac{s_1 + s_2}{2} \approx 23^{\text{h}}.$$

The hour angle of the star setting is

$$t_\star = s_c - s_1 = 23.1^{\text{h}} - 20.06^{\text{h}} = 3.04^{\text{h}}.$$

Therefore, we may estimate the declination **[2 pt]** by the formula (2):

$$\cos t_\star = -\tan\varphi\tan\delta_\star \quad\Rightarrow\quad \tan\delta_\star = -\frac{\cos t_\star}{\tan\varphi} = -0.587 \quad\Rightarrow\quad \delta_\star \approx -30^\circ.$$

These coordinates are quite similar to the coordinates of Fomalhaut (α Piscis Austrinus; $\alpha = 22^{\text{h}}\,57^{\text{m}};\ \delta = -29^\circ\,37'$) **[1 pt].**

6 Space Vodka

The table below shows the characteristics of two space masers.
Which source is larger in size and how many times larger?

	Water H_2O	Methanol CH_3OH
Wavelength, cm	1.35	4.5
Flux density, kJy	1500	0.15
Brightness temperature, $\log T$ [K]	17	6.5
Parallax, mas	2.5	0.75

Solution:

An astrophysical maser is a naturally occurring source of stimulated spectral line emission. Brightness temperature is the temperature at which a black body *would have to be* in order to duplicate the observed intensity in a spectral line; it is not the real thermodynamic temperature of some body!

Distance d to the radio source is inversely proportional to its parallax ϖ: $d \propto \varpi^{-1}$ **[2 pt].**

Flux density $S_\nu = B_\nu \Omega$ **[2 pt],** where Ω is the solid angle corresponding to the source, $\Omega \propto R^2 d^{-2}$ **[2 pt].** R is the spatial size of the source. B_ν is spectral radiance which depends on the brightness temperature T and emission wavelength λ. As brightness temperature corresponds to some black body and energy of photons in radio is significantly lower than thermal energy of the matter ($hc/\lambda \ll k_B T$), B_ν can be expressed via the Rayleigh–Jeans law[5] **[2 pt]:**

$$B_\nu = \frac{2k_B T \nu^2}{c^2} = \frac{2k_B T}{\lambda^2} \propto T\lambda^{-2}.$$

Therefore, the size of the source

$$R \propto \sqrt{\Omega d^2} \propto d\sqrt{\frac{S_\nu}{B_\nu}} \propto \frac{1}{\varpi}\sqrt{\frac{S_\nu}{T\lambda^{-2}}} \propto \frac{\lambda}{\varpi}\sqrt{\frac{S_\nu}{T}}.$$

The size ratio is

$$\frac{R_2}{R_1} = \frac{\lambda_2}{\lambda_1} \cdot \frac{\varpi_1}{\varpi_2} \cdot \sqrt{\frac{S_{\nu,2}}{S_{\nu,1}} \cdot \frac{T_1}{T_2}} = \frac{4.5}{1.35} \times \frac{2.5}{0.75} \times \sqrt{\frac{0.15}{1500} \times \frac{10^{17}}{10^{6.5}}} \approx 2 \cdot 10^4.$$

So, the methanol maser is much bigger than the water one **[2 pt].**

[5] The flux density is measured in *janskys,* essential non-SI units excessively used in radio astronomy; 1 Jy $= 10^{-26}$ (W/m^2)/Hz. That is why in this problem, the radiation laws should be in terms of frequencies. If one uses B_λ instead of B_ν, the result changes several times, just as blackbody emission peaks at wavelength $\lambda_{\max}$, or at frequency $\nu_{\max} \neq c/\lambda_{\max}$.

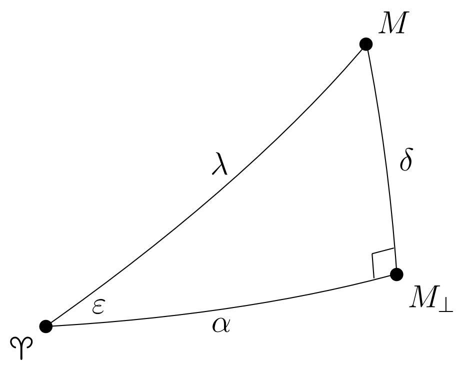
7 One Thousand and One Nights

On a rocky planet in a circular orbit around a star, equatorial and ecliptic coordinates are arranged in the same way as on the Earth. The local ecliptic passes through points with equatorial coordinates ($\alpha_1 = 20.4°$; $\delta_1 = 22.5°$) and ($\alpha_2 = 74.7°$; $\delta_2 = 49.0°$). Calculate the fraction of the planet's surface where polar nights can occur. Neglect the atmosphere.

Solution:

First, we determine the inclination ε of the ecliptic to the equator as the inclination of a great circle passing through the points $(\alpha_1; \delta_1)$, $(\alpha_2; \delta_2)$.

Both the right ascension and the ecliptic longitude are counted from the local point of the vernal equinox $(0; 0)$ — that is the point of intersection of the ecliptic and the celestial equator. Therefore, the inclination can be determined from a spherical triangle with vertices at the vernal equinox point ♈, a point on the ecliptic M and the projection of this point on the celestial equator $M_\perp$.



Using the five-part rule and the law of sines we obtain:

$$\sin\lambda\cos\varepsilon = \sin\alpha\cos\delta - \cos\alpha\sin\delta\cos 90° \quad \Rightarrow \quad \sin\lambda = \frac{\sin\alpha\cos\delta}{\cos\varepsilon},$$

$$\frac{\sin\delta}{\sin\varepsilon} = \frac{\sin\lambda}{\sin 90°}.$$

Substitution of $\sin\lambda$ yields

$$\frac{\sin\delta}{\sin\varepsilon} = \frac{\sin\alpha\cos\delta}{\cos\varepsilon} \quad \Rightarrow \quad \tan\delta = \sin\alpha\tan\varepsilon.$$

Next, we can get an estimate of the inclination from the data on any of the two points,

$$\varepsilon = \arctan\frac{\tan\delta_1}{\sin\alpha_1} = \arctan\frac{\tan 22.5°}{\sin 20.4°} = \arctan 1.19 = 50°.$$

Another approach. We introduce a Cartesian coordinate system centered at the place of observation. The x-axis is directed to the local point of the vernal equinox, the z-axis is directed to the north celestial pole, the y-axis complements the axes to a right-handed coordinate system. The vectors of two points are as follows:

$$\vec{v}_1 = \begin{pmatrix} \cos\delta_1 \cos\alpha_1 \\ \cos\delta_1 \sin\alpha_1 \\ \sin\delta_1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} \cos\delta_2 \cos\alpha_2 \\ \cos\delta_2 \sin\alpha_2 \\ \sin\delta_2 \end{pmatrix}.$$

Next, we determine the vector $\vec{w}$ of the pole of the great circle (namely, the north ecliptic pole) using the vector product

$$\vec{w} = \frac{\vec{v}_1 \times \vec{v}_2}{|\vec{v}_1 \times \vec{v}_2|}$$

where

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\delta_1 \cos\alpha_1 & \cos\delta_1 \sin\alpha_1 & \sin\delta_1 \\ \cos\delta_2 \cos\alpha_2 & \cos\delta_2 \sin\alpha_2 & \sin\delta_2 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.866 & 0.322 & 0.383 \\ 0.173 & 0.633 & 0.755 \end{vmatrix},$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} 0.322 \times 0.755 - 0.633 \times 0.383 \\ 0.173 \times 0.383 - 0.866 \times 0.755 \\ 0.866 \times 0.633 - 0.173 \times 0.322 \end{pmatrix} = \begin{pmatrix} 0.000671 \\ -0.587571 \\ 0.492472 \end{pmatrix},$$

$$|\vec{v}_1 \times \vec{v}_2| = \sqrt{0.000671^2 + (-0.587571)^2 + 0.492472^2} = 0.767,$$

$$\vec{w} = \begin{pmatrix} 0.0009 \\ -0.7664 \\ 0.6424 \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\alpha_0 \\ \cos\theta \sin\alpha_0 \\ \sin\theta \end{pmatrix}.$$

$\vec{w}$ points to the north ecliptic pole. $\theta = 90^\circ - \varepsilon$ and α_0 are the declination and the right ascension of the north ecliptic pole respectively,

$$\theta = \arcsin w_z = 40^\circ \quad \Rightarrow \quad \varepsilon = \arccos w_z = 50^\circ.$$

Polar nights are possible only at latitudes with a modulus greater than $\varphi_0 = 90^\circ - \varepsilon = \theta = 40^\circ$. The corresponding points form two spherical caps around the poles. Let us determine the fraction of the planet's surface with latitudes modulus more than 40°:

$$\eta = \frac{2S_{\text{cap}}}{S_{\text{sphere}}} = \frac{2 \cdot (2\pi R h)}{4\pi R^2}$$

where

$$h = R[1 - \cos(90^\circ - \varphi_0)] = R(1 - \sin\theta)$$

is the height of the spherical cap.

Finally,

$$\eta = \frac{2 \times 2\pi R^2(1 - \sin\theta)}{4\pi R^2} = 1 - \sin 40^\circ = 0.357 = 35.7\,\%.$$

Marking Scheme:

- Inclination of the ecliptic.

  In case of using spherical trigonometry:

  - Distance between two points on a sphere — **2 pt.**
  - Choosing a valid spherical triangle — **2 pt.**
  - Inclination ε — **1 pt.**

  It is acceptable to write the great circle equation for the ecliptic in polar coordinates without derivation.

  In case of using Cartesian coordinates:

  - Conversion from ecliptic coordinates to Cartesian coordinates: formula + evaluation — **1 pt + 1 pt.**
  - Vector multiplication — **2 pt.**
  - Inclination ε — **1 pt.**

- Fraction of the surface with polar nights:

  - Limiting latitudes ($|\varphi| = 90^\circ - \varepsilon$) — **2 pt.**
  - Area of spherical caps — **2 pt.**
  - Fraction η — **1 pt.**

8 Omega Sirius

Estimate the distance between α and ω Canis Majoris in parsecs.

	α CMa	ω CMa
Right ascension	$06^{h}\,45^{m}\,08.917^{s}$	$07^{h}\,14^{m}\,48.653^{s}$
Declination	$-16^\circ\,42'\,58.02''$	$-26^\circ\,46'\,21.61''$
Apparent magnitude in V band	-1.46	3.82
Spectral type	A1Vm	B2.5Ve
Bolometric correction	0	-2.2
Mass, $\mathfrak{M}_\odot$	2.063 ± 0.023	10.1 ± 0.7

Solution:

Using spherical law of cosines we obtain the angular distance between the stars:

$$\cos\theta = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2),$$
$$\cos\theta = \sin(-16.72^\circ)\sin(-26.77^\circ) + \cos(-16.72^\circ)\cos(-26.77^\circ)\cos\left(29.66^{m} \times \frac{1^\circ}{4^{m}}\right) = 0.9775,$$
$$\theta = \arccos 0.9775 = 12.2^\circ.$$

Both stars are main sequence stars — luminosity class V, so we can assume for such masses there is a power law relationship between mass $\mathfrak{M}$ and luminosity L:

$$\frac{L}{L_\odot} = 1.5\left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^{3.5}.$$

Comparing the stars with the Sun ($M_\odot = 4.74^{m}$) we obtain absolute bolometric magnitudes of them:

$$M = M_\odot - 2.5\log\frac{L}{L_\odot} = M_\odot - 2.5\log\left(1.5\left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^{3.5}\right);$$
$$M_1 = 4.74 - 2.5\log\left(1.5 \cdot 2.063^{3.5}\right) = 1.5^{m},$$
$$M_2 = 4.74 - 2.5\left(1.5 \cdot 10.1^{3.5}\right) = -4.5^{m}.$$

Absolute magnitudes in V band are

$$M_{V,1} = M_1 = 1.5^{m},$$
$$M_{V,2} = M_2 - BC = -4.5 + 2.2 = -2.3^{m}.$$

Now we can estimate distances to the stars from known absolute M_V and apparent m_V visual magnitudes:

$$M_V = m_V + 5 - 5\log d \quad \Rightarrow \quad d = 10^{0.2(m_V - M_V + 5)};$$

$$d_1 = 10^{0.2\times(-1.46-1.5+5)} = 2.6 \text{ pc},$$
$$d_2 = 10^{0.2\times(3.82+2.3+5)} = 168 \text{ pc}.$$

The obtained values are not very accurate due to the approximations made.

Finally, the distance between stars is

$$d = \sqrt{d_1^2 + d_2^2 - 2d_1d_2\cos\theta} \approx 166 \text{ pc}.$$

Actually, since $d_2 \gg d_1$, we could conclude immediately that $d \approx d_2$.

Note. In the solution above ω CMa turned out to be noticeably fainter than in reality. This is mainly due to the fact that the mass–luminosity relation for massive stars does not provide an accurate estimate of luminosity. Due to the significant scatter in the mass-luminosity diagram for massive stars, it is acceptable to use a proportionality $L \propto \mathfrak{M}^4$, which results in a final estimate of the distance between the stars of ~ 240 pc.

Marking Scheme:

- Angular distance between the stars — **3 pt.**
- Absolute bolometric magnitudes estimation: $L \propto \mathfrak{M}^\alpha$
  - $\alpha \in [3.5;\ 4.0]$ — **2 pt.**
  - $\alpha \in [3.0;\ 3.5)$ — 1 pt.
- Absolute visual magnitudes — **1 pt.**
- Distance to each star — **2 pt.**
- Distance between the stars — **2 pt.**

9 Get Into the Loop

An artificial satellite moves in an orbit with eccentricity $e > 0$, semi-major axis a, and inclination $0 < i < 90^\circ$. The argument of perigee $\omega = 0^\circ$. Assume the Earth to be an ideal sphere rotating at a constant angular velocity W.

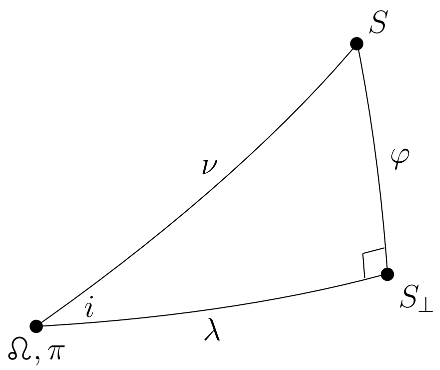
a) What are the satellite's geographic latitude $\varphi(\nu)$ and longitude $\lambda(\nu)$ depending on the true anomaly ν?

b) Consider a satellite in a geosynchronous orbit ($T = 23^\mathrm{h}\,56^\mathrm{m}\,04^\mathrm{s}$) with $e = 0.30$ and $i = 1.00$ rad. Calculate the ground track of the satellite (projection of the trajectory onto the surface of the rotating Earth) and draw it. For convenience, use the answer sheet with table and graph grid.

Hint: $$\int \frac{dx}{(1 + a\cos x)^2} = \frac{2\arctan\left(\sqrt{\frac{1-a}{1+a}} \cdot \tan\frac{x}{2}\right)}{(1-a^2)^{3/2}} - \frac{a\sin x}{(1-a^2)(1+a\cos x)} + \text{const.}$$

Solution:

a) Let the origin point be the ascending node of the satellite's orbit — in this problem it coincides with perigee ($\omega = 0^\circ$). For the respective spherical triangle:



$$\frac{\sin\varphi}{\sin i} = \frac{\sin\nu}{\sin 90^\circ} \qquad \Rightarrow \qquad \sin\varphi = \sin i \sin\nu,$$

$$\cos i = \frac{\tan\lambda}{\tan\nu} \qquad \Rightarrow \qquad \tan\lambda = \cos i \tan\nu.$$

Taking into account the rotation of the Earth, we may put down the geographical coordinates of the satellite:

$$\varphi' = \varphi = \arcsin(\sin i \sin\nu),$$
$$\lambda' = \lambda - W\int_0^t dt. \tag{3}$$

Due to rotational symmetry, the longitude of the satellite is determined up to a shift by an arbitrary constant value.

The angular momentum of the satellite is conserved,

$$l = \frac{d\nu}{dt} r^2 = \sqrt{G\mathfrak{M}_\oplus \, a(1-e^2)} = \text{const.}$$

At the same time, the geocentric distance of the satellite depends on its true anomaly:

$$r = \frac{a(1-e^2)}{1+e\cos\nu}.$$

Therefore,

$$dt = \frac{r^2 \, d\nu}{\sqrt{G\mathfrak{M}_\oplus \, a(1-e^2)}} = \sqrt{\frac{a^3(1-e^2)^3}{GM_\oplus}} \frac{d\nu}{(1+e\cos\nu)^2}.$$

Next, we can replace the integration variable in (3):

$$\lambda' = \lambda - W\sqrt{\frac{a^3(1-e^2)^3}{GM_\oplus}} \int\limits_0^\nu \frac{d\nu}{(1+e\cos\nu)^2} =$$

$$= \lambda - W\sqrt{\frac{a^3}{GM_\oplus}}(1-e^2)^{3/2} \left[\frac{2\arctan\left(\sqrt{\frac{1-e}{1+e}} \cdot \tan\frac{\nu}{2}\right)}{(1-e^2)^{3/2}} - \frac{e\sin\nu}{(1-e^2)(1+e\cos\nu)} \right] =$$

$$= \arctan(\cos i \tan\nu) - W\sqrt{\frac{a^3}{GM_\oplus}} \left[2\arctan\left(\sqrt{\frac{1-e}{1+e}} \cdot \tan\frac{\nu}{2}\right) - \frac{e\sin\nu\sqrt{1-e^2}}{1+e\cos\nu} \right].$$

b) For geosynchronous orbit

$$\frac{2\pi}{T} = W = \sqrt{\frac{G\mathfrak{M}_\oplus}{a^3}},$$

so we can simplify the expression above a bit:

$$\lambda' = \arctan(\cos i \tan\nu) + \frac{e\sin\nu\sqrt{1-e^2}}{1+e\cos\nu} - 2\arctan\left(\sqrt{\frac{1-e}{1+e}} \cdot \tan\frac{\nu}{2}\right).$$

Thoughts on plotting. It is pretty obvious that the relative motion of the satellite is periodical and the track is closed. To plot the graph, we have to calculate φ' and λ' values for ν in range from $-\pi$ to π. In fact, the work can be simplified by noting that

$$\varphi'(-\nu) = -\varphi(\nu), \quad \lambda'(-\nu) = -\lambda'(\nu),$$

so it is enough to probe the range of ν from 0 to π, and then mirror the results.

Also, note that arctan(# tan ν) is ambiguous. λ' follows the motion of the satellite, therefore arctan should be re-defined to become a continuous function for $\nu \in [0;\ \pi)$:

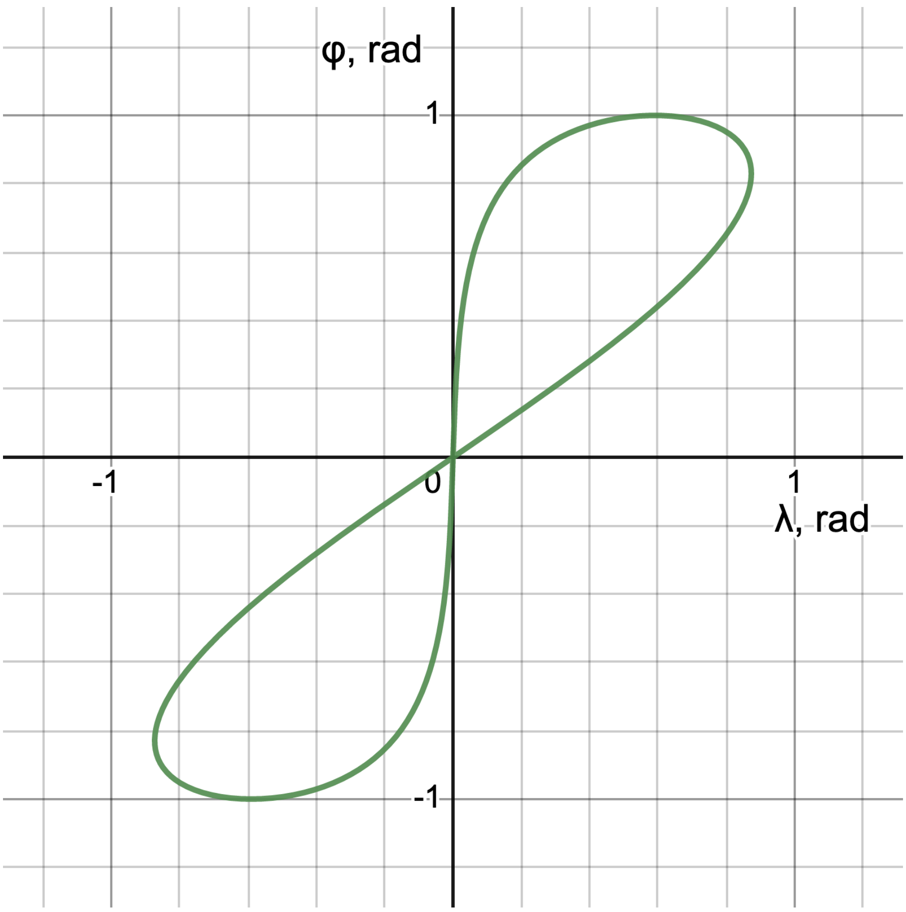
$$\arctan(\cos i \tan \nu) := \begin{cases} \underline{\arctan}(\cos i \tan \nu), & \nu \leqslant \pi/2, \\ \underline{\arctan}(\cos i \tan \nu) + \pi, & \nu > \pi/2. \end{cases}$$

Here $\underline{\arctan}$ is "conventional" inverse trigonometric function ranged from $-\pi/2$ to $\pi/2$. Of course, $\underline{\arctan}(+\infty) := \pi/2$. Such a trick does not affect the parts of the expression that came along with that *remarkable* integral, since $\nu/2$ appears there as an argument.

For calculations, it is convenient to substitute the values of e and i and regroup:

$$\lambda'(\nu) \approx \begin{cases} \arctan(0.54 \tan \nu) + \dfrac{0.286 \cdot \sin \nu}{1 + 0.3 \cos \nu} - 2 \arctan\left(0.734 \tan \frac{\nu}{2}\right), & \nu \in [0;\ \pi/2), \\ \arctan(0.54 \tan \nu) + 3.142 + \dfrac{0.286 \cdot \sin \nu}{1 + 0.3 \cos \nu} - 2 \arctan\left(0.734 \tan \frac{\nu}{2}\right), & \nu \in (\pi/2;\ \pi), \\ -\lambda'(-\nu), & \nu \in (-\pi;\ 0) \setminus \{-\pi/2\}. \end{cases}$$

Finally, here is the plot we need:



Marking Scheme:

- Expression for latitude φ — **3 pt.**
- Expression for longitude λ — **3 pt.**
- Angular momentum conservation — **3 pt.**
- Accounting the rotation of the Earth, like (3) — **3 pt.**
- Expression for longitude λ' taking into account rotation — **3 pt.**
- Angular velocity of a geosynchronous satellite — **2 pt.**
- Plot of the ground track — **3 pt.**

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Hubble constant	$H_0 = 70$ (km/s)/Mpc
Astronomical unit	1 au = $149.6 \cdot 10^6$ km
Parsec	1 pc = 206 265 au

Earth

Radius	$R_\oplus = 6371$ km
Obliquity	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^\mathrm{d}$
Orbital eccentricity	$e_\oplus = 0.0167$

Moon

Radius	$R_☾ = 1737$ km
Orbital period	$T_☾ = 27.32^\mathrm{d}$
Orbital inclination	$i_☾ = 5.1^\circ$

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Absolute magnitude	$M_\odot = 4.74^\mathrm{m}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Emission constants

Stefan–Boltzmann	$\sigma = 5.67 \cdot 10^{-8}$ (W/m^2)/K^4
Wien's displacement	$b = 2898$ µm · K

UBV system	Mean wavelengths
U band	$\lambda_U = 364$ nm
B band	$\lambda_B = 442$ nm
V band	$\lambda_V = 540$ nm

Hydrogen spectrum

Lyman Lα	$\lambda_{L\alpha} = 1215.7$ Å
Balmer Hα	$\lambda_{H\alpha} = 6562.8$ Å