1 Scattering of Stones

There are various spectral types of asteroids. Multiband photometry data can be used to determine the type of a particular asteroid. The article by Choi et al. (2023) provides photometric data (g, r, i, z) for several thousand asteroids, as well as describes a method for determining their types. The catalog of photometric data is provided in a separate table `ast.dat`. The description of the columns is indicated in the file itself.

We will use a simplified version of the method. First of all, the standardized magnitudes are transformed to the reflectances in magnitude by subtracting the solar colors:

$$(g'-r') = (g-r) - (g-r)_\odot\,, \qquad (g-r)_\odot = 0.61;$$
$$(r'-i') = (r-i) - (r-i)_\odot\,, \qquad (r-i)_\odot = 0.35;$$
$$(i'-z') = (i-z) - (i-z)_\odot\,, \qquad (i-z)_\odot = 0.16.$$

Next, the coefficient a of least-squares linear regression is estimated:

$$(r'-i') = a \cdot (g'-r') + b.$$

We assume that the errors in both color indexes are distributed according to the same law, so we use the simple estimation — Deming linear regression:

$$\overline{(g'-r')} = \frac{1}{n}\sum_{j=1}^{n}(g'-r')_j\,, \qquad \overline{(r'-i')} = \frac{1}{n}\sum_{j=1}^{n}(r'-i')_j\,;$$
$$s_{g'-r'} = \frac{1}{n}\sum_{j=1}^{n}\left((g'-r')_j - \overline{(g'-r')}\right)^2, \qquad s_{r'-i'} = \frac{1}{n}\sum_{j=1}^{n}\left((r'-i')_j - \overline{(r'-i')}\right)^2;$$
$$s_{g'-r',r'-i'} = \frac{1}{n}\sum_{j=1}^{n}\left((g'-r')_j - \overline{(g'-r')}\right)\left((r'-i')_j - \overline{(r'-i')}\right);$$
$$a = \frac{s_{r'-i'} - s_{g'-r'} + \sqrt{(s_{r'-i'} - s_{g'-r'})^2 + 4s^2_{g'-r',r'-i'}}}{2s_{g'-r',r'-i'}}.$$

Further, the so-called principal component (PC) is estimated for each asteroid. For the j-th asteroid

$$\mathrm{PC}_j = (g'-r')_j\cos\theta + (r'-i')_j\sin\theta, \qquad \theta \equiv \arctan a.$$

Based on the values of PC and $(i'-z')$, the spectral types of asteroids can be approximately determined. The taxonomy boundaries are given in Table 1 below.

We will only work with asteroids whose magnitude uncertainties are less than or equal to 0.05 in all photometric bands.

a) Plot the dependence $(r' - i')$ on $(g' - r')$ for suitable asteroids. How many objects are on the graph?

b) Determine the coefficient a of Deming linear regression in the $(g' - r')$ versus $(r' - i')$ color plane.

c) Plot the dependence $(i' - z')$ on PC for the same asteroids.

d) What spectral type does asteroid 8 Flora belong to?

e) How many asteroids belong to each type listed in the table?

Table 1: Parametric boundaries for different spectral types of asteroids

Type	Boundaries
L & D	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.150 \leqslant (i' - z') < 0.040$
S	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.320 \leqslant (i' - z') < -0.150$
V	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.610 \leqslant (i' - z') < -0.320$
C	$\mathrm{PC} \geqslant -0.395$ $(i' - z') + 2.778\mathrm{PC} < -0.864$ $-0.210 \leqslant (i' - z') < 0.040$
X	$\mathrm{PC} < -0.195$ $(i' - z') + 2.778\mathrm{PC} \geqslant -0.864$ $-0.210 \leqslant (i' - z') < 0.040$

2 Support Service

An analemma (from Ancient Greek αναλημμα – "support") is a diagram showing the position of the Sun in the sky as seen from a fixed location on the Earth at the same mean solar time, as that position varies over the course of a year. It is directly related to the so-called equation of time — the difference between mean and true solar time (that is, the difference between the right ascensions of the true and mean Sun):

$$\eta = t_{\text{mean}} - t_{\text{true}} = \alpha_{\text{true}} - \alpha_{\text{mean}}.$$

This difference is due to two reasons: the tilt of the planet's axis of rotation (obliquity) and the eccentricity of its orbit.

Assume the planet pass perihelion at the moment $t = 0$. If the planet moved uniformly, its position would be determined by the mean anomaly

$$M(t) = \frac{2\pi t}{T},$$

where T is the orbital period of the planet.

The actual position of the planet in its orbit is determined by the true anomaly ν. The true anomaly ν is geometrically related to the eccentric anomaly E, which shows the position of some imaginary point on the auxiliary circle (see Figure 1).

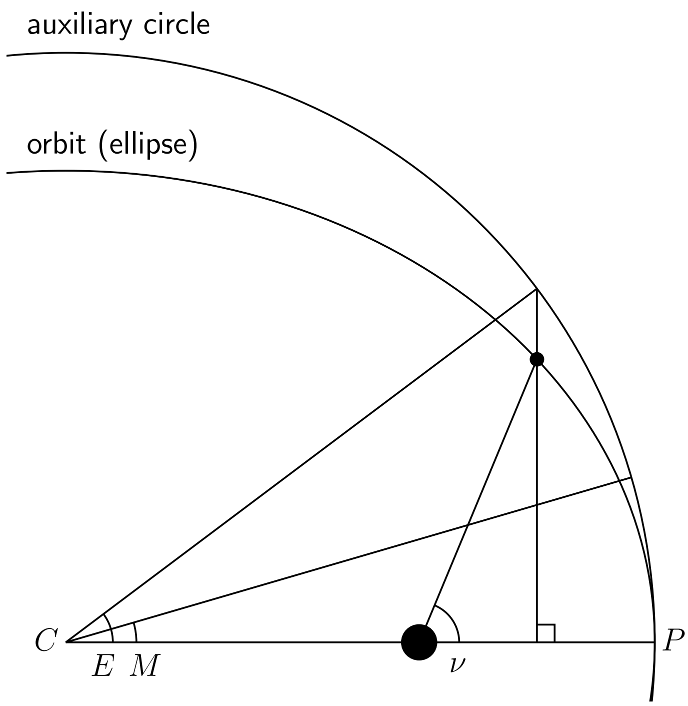

Figure 1: Geometric meaning and connection between different anomalies.
C is the center of the ellipse and auxiliary circle, P is the pericenter of the orbit

Please note that

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2},$$

where e is the eccentricity of the planet's orbit. Also, the mean and eccentric anomalies (expressed in radians) are related by Kepler's equation:

$$E - e \cdot \sin E = M.$$

a) Plot the difference $(\nu - M)$ as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars. For convenience, express $(\nu - M)$ in "local" minutes: 1 solar day = 24 hours, 1 hour = 60 minutes.

Let x be the ecliptic longitude of the Sun measured from the vernal equinox for the planet when this planet is at perihelion. You can find the corresponding values in the Constants Table.

b) Express the "local" ecliptic longitude of the Sun λ in terms of ν and x.

Hour angles and right ascensions are measured along the celestial equator. Since the ecliptic is inclined to the celestial equator by an angle of ε, both of these parameters are related to the ecliptic longitude nonlinearly. For example, for right ascension α we have:

$$\tan\alpha = \tan\lambda \cdot \cos\varepsilon.$$

c) Plot the difference $(\alpha - \lambda)$ expressed in "local" minutes as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars.

The equation of time can be represented as the sum

$$\eta(t) = (\nu - M) + (\alpha - \lambda).$$

d) Plot the equation of time η expressed in "local" minutes as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars.

The declination of the Sun δ can be expressed by the formula

$$\sin\delta = \sin\lambda \cdot \sin\varepsilon.$$

e) Plot the analemma corresponding to mean noon for an observer at mid-latitudes of the Northern Hemisphere of the Earth and Mars (north is up, west is to the right).

f) There are the analemmas of five imaginary planets in Figure 2. Determine the parameters e, x, and ε for each model. In case there are several possible answers, please provide any.

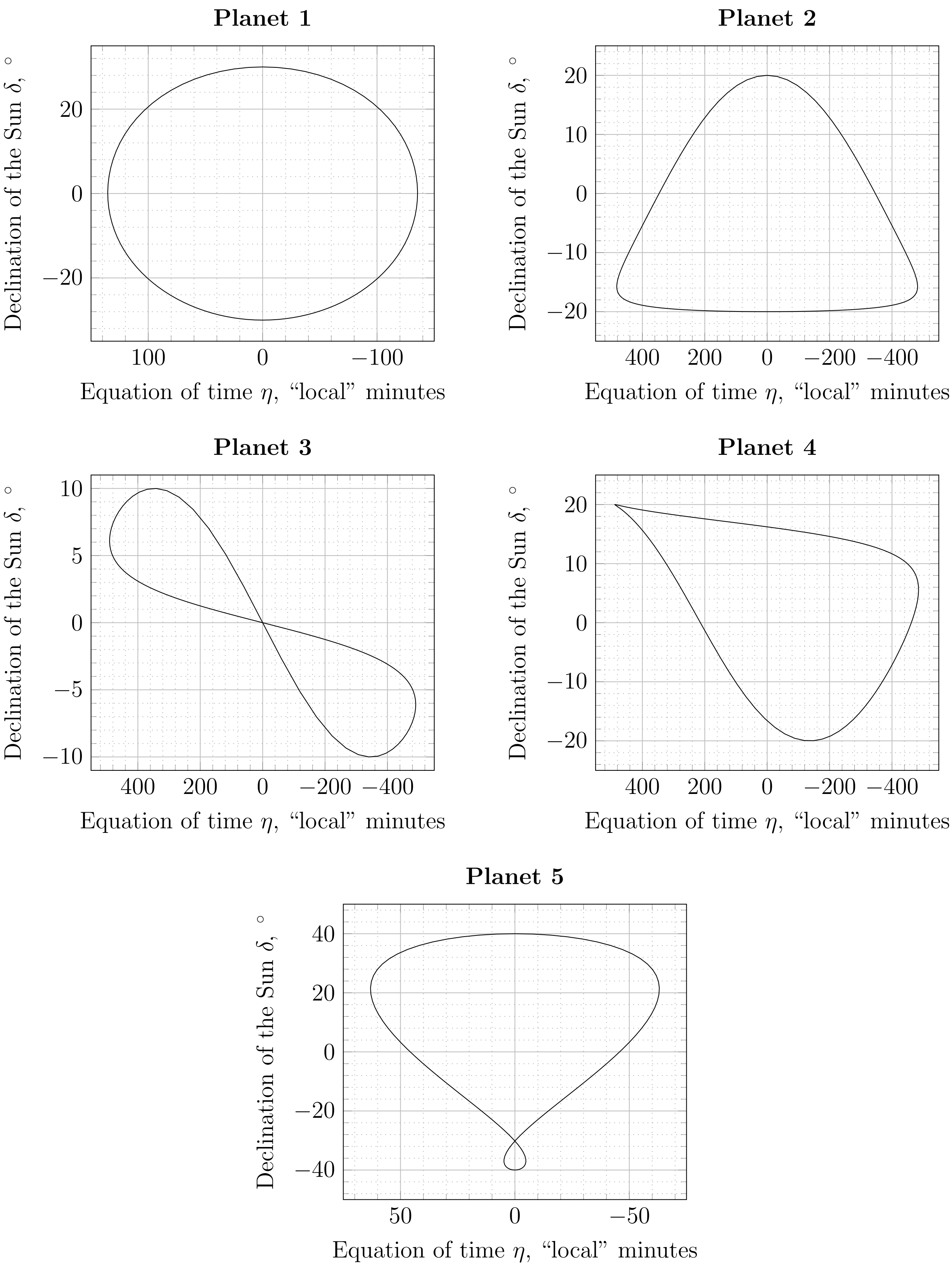

Figure 2: Analemmas of five imaginary planets

3 Ephemeric Quest

The file `ephem.dat` contains ephemerides of some small body of the Solar system. The first and second columns of the file contain the date and universal time of the observation, followed by the right ascension (in hours, minutes and seconds) and declination (in degrees, minutes and seconds) for the observer on the Earth (geocentric location):

Date	Time	Right ascension	Declination
2023-Mar-18	00:00:00	$23^{\mathrm{h}}\,59^{\mathrm{m}}\,03.88^{\mathrm{s}}$	$0^\circ\,44'\,46.9''$
2023-Mar-19	00:00:00	$0^{\mathrm{h}}\,0^{\mathrm{m}}\,41.94^{\mathrm{s}}$	$0^\circ\,55'\,34.0''$
...	...	...	...

The absolute magnitude of the object is $H = 18.51^{\mathrm{m}}$. The orbit of the object is circular, the inclination is small.

a) Determine whether the object's orbit lies inside or outside the Earth's orbit.

b) Estimate the radius of the object's orbit.

c) Estimate the minimal time interval between two consecutive greatest elongations or consecutive quadratures.

d) Determine the difference between the maximal and minimal apparent magnitudes of the object for an observer on the Earth during one synodic period.

e) Plot the dependence of the apparent magnitude of the object on time for an observer on the Earth during one synodic period.

Hint: for the Solar system objects, the absolute magnitude, commonly called H, is defined as the apparent magnitude that the object would have if it were one astronomical unit (1.00 au) from both the Sun and the observer, and in conditions of ideal solar opposition (the apparent phase is 1.00).

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R} = 8.314$ J/(mol · K)
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au = $149.6 \cdot 10^6$ km
Parsec	1 pc = 206 265 au
Hubble constant	$H_0 = 70$ (km/s)/Mpc

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Mass	$\mathfrak{M}_\odot = 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot = 4.74^\mathrm{m}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Earth

Radius	$R_\oplus = 6371$ km
Sidereal day	$\tau_\oplus = 23^\mathrm{h}\,56^\mathrm{m}\,04^\mathrm{s}$
Obliquity of ecliptic	$\varepsilon_\oplus = 23.437°$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^\mathrm{d}$
Orbital eccentricity	$e_\oplus = 0.0167$
	$x_\oplus = 283.3°$

Mars

Radius	$R_M = 3390$ km
Sidereal day	$\tau_M = 24^\mathrm{h}\,37^\mathrm{m}\,23^\mathrm{s}$
Obliquity (axial tilt)	$\varepsilon_M = 25.192°$
Orbital semi-major axis	$a_M = 1.52$ au
Orbital period	$T_M = 1.88$ yr
Orbital eccentricity	$e_M = 0.0934$
	$x_M = 251°$