1 Scattering of Stones

There are various spectral types of asteroids. Multiband photometry data can be used to determine the type of a particular asteroid. The article by Choi et al. (2023) provides photometric data (g, r, i, z) for several thousand asteroids, as well as describes a method for determining their types. The catalog of photometric data is provided in a separate table `ast.dat`. The description of the columns is indicated in the file itself.

We will use a simplified version of the method. First of all, the standardized magnitudes are transformed to the reflectances in magnitude by subtracting the solar colors:

$$(g'-r') = (g-r) - (g-r)_\odot\,, \qquad (g-r)_\odot = 0.61;$$
$$(r'-i') = (r-i) - (r-i)_\odot\,, \qquad (r-i)_\odot = 0.35;$$
$$(i'-z') = (i-z) - (i-z)_\odot\,, \qquad (i-z)_\odot = 0.16.$$

Next, the coefficient a of least-squares linear regression is estimated:

$$(r'-i') = a\cdot(g'-r') + b.$$

We assume that the errors in both color indexes are distributed according to the same law, so we use the simple estimation — Deming linear regression:

$$\overline{(g'-r')} = \frac{1}{n}\sum_{j=1}^{n}(g'-r')_j\,, \qquad \overline{(r'-i')} = \frac{1}{n}\sum_{j=1}^{n}(r'-i')_j\,;$$
$$s_{g'-r'} = \frac{1}{n}\sum_{j=1}^{n}\left((g'-r')_j - \overline{(g'-r')}\right)^2, \qquad s_{r'-i'} = \frac{1}{n}\sum_{j=1}^{n}\left((r'-i')_j - \overline{(r'-i')}\right)^2;$$
$$s_{g'-r',r'-i'} = \frac{1}{n}\sum_{j=1}^{n}\left((g'-r')_j - \overline{(g'-r')}\right)\left((r'-i')_j - \overline{(r'-i')}\right);$$
$$a = \frac{s_{r'-i'} - s_{g'-r'} + \sqrt{(s_{r'-i'} - s_{g'-r'})^2 + 4s^2_{g'-r',r'-i'}}}{2s_{g'-r',r'-i'}}.$$

Further, the so-called principal component (PC) is estimated for each asteroid. For the j-th asteroid

$$\mathrm{PC}_j = (g'-r')_j\cos\theta + (r'-i')_j\sin\theta, \qquad \theta\equiv\arctan a.$$

Based on the values of PC and $(i'-z')$, the spectral types of asteroids can be approximately determined. The taxonomy boundaries are given in Table 1 below.

We will only work with asteroids whose magnitude uncertainties are less than or equal to 0.05 in all photometric bands.

a) Plot the dependence $(r' - i')$ on $(g' - r')$ for suitable asteroids. How many objects are on the graph?

b) Determine the coefficient a of Deming linear regression in the $(g' - r')$ versus $(r' - i')$ color plane.

c) Plot the dependence $(i' - z')$ on PC for the same asteroids.

d) What spectral type does asteroid 8 Flora belong to?

e) How many asteroids belong to each type listed in the table?

Table 1: Parametric boundaries for different spectral types of asteroids

Type	Boundaries
L & D	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.150 \leqslant (i' - z') < 0.040$
S	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.320 \leqslant (i' - z') < -0.150$
V	$-0.195 \leqslant \mathrm{PC} < 0.025$ $-0.610 \leqslant (i' - z') < -0.320$
C	$\mathrm{PC} \geqslant -0.395$ $(i' - z') + 2.778\mathrm{PC} < -0.864$ $-0.210 \leqslant (i' - z') < 0.040$
X	$\mathrm{PC} < -0.195$ $(i' - z') + 2.778\mathrm{PC} \geqslant -0.864$ $-0.210 \leqslant (i' - z') < 0.040$

Solution:

a) The original catalog lists 6793 asteroids. We leave only objects with the magnitude uncertainties less than or equal to 0.05^m in all photometric bands, as prescribed. The filtered catalog contains $n = 3504$ suitable objects.

The dependence of $(r' - i')$ on $(g' - r')$ is shown in Figure 1.

b) First, we list the values of all calculated quantities needed for estimating the coefficient a.

$$\overline{(g' - r')} = \frac{1}{3504} \sum_{j=1}^{3504} (g' - r')_j = -0.100;$$

$$\overline{(r' - i')} = \frac{1}{3504} \sum_{j=1}^{3504} (r' - i')_j = -0.177;$$

$$s_{g'-r'} = \frac{1}{3504} \sum_{j=1}^{3504} \left((g' - r')_j + 0.100\right)^2 = 9.36 \cdot 10^{-3};$$

$$s_{r'-i'} = \frac{1}{3504}\sum_{j=1}^{3504}\big((r'-i')_j + 0.177\big)^2 = 3.99\cdot 10^{-3};$$

$$s_{g'-r',r'-i'} = \frac{1}{3504}\sum_{j=1}^{3504}\big((g'-r')_j + 0.100\big)\big((r'-i')_j + 0.177\big) = 2.56\cdot 10^{-3}.$$

Next, we calculate an estimate of the angular regression coefficient

$$a = \frac{3.99 - 9.36 + \sqrt{(3.99-9.36)^2 + 4\cdot(2.56)^2}}{2\cdot 2.56} = 0.400;$$

$$\theta = \arctan(0.400) = 21.8^\circ.$$

Note that the simple least squares method, assuming errors in only one of the coordinates, leads to a significantly different result and is not applicable for this problem.

c) Using the value of Deming regression angle θ obtained above, we calculate the principal component for each asteroid:

$$\mathrm{PC}_j = \cos 21.8^\circ\cdot(g'-r')_j + \sin 21.8^\circ\cdot(r'-i')_j = 0.928(g'-r')_j + 0.371(r'-i')_j\,.$$

The dependence of $(i'-z')$ on PC is shown in Figure 2.

d) For 8 Flora $(i'-z') = -0.19$, PC $= 2.2\cdot 10^{-3}$. This asteroid belongs to S-type (silicaceous).

e) Due to rounding errors in the previous stages, the number of asteroids may slightly differ from the one shown in the table.

Type	L & D	S	V	C	X	Other
N	606	1439	207	582	401	269
%	17	41	6	17	11	8

The spectral types in question approximately correspond to SMASS II taxonomic system. Note that in the filtered catalog, the most numerous were the S-type (silicaceous) asteroids; approximately 17 % of the real asteroids are of this type. In reality, among all known asteroids, the most numerous objects (about 77 %) are the C-type asteroids (carbonaceous), but they comprise only 17 % in the catalog in this problem. The V-type (volcanic-type or vestoids) asteroids were the least common in our sample, in reality they also comprise approximately 6 % of the main-belt asteroids. The X-group collects together several types of asteroids with similar spectra, namely metallic asteroids, ones with high albedo and asteroids with red spectra. L & D-type correspond to the asteroids with very steep red slope in spectra.

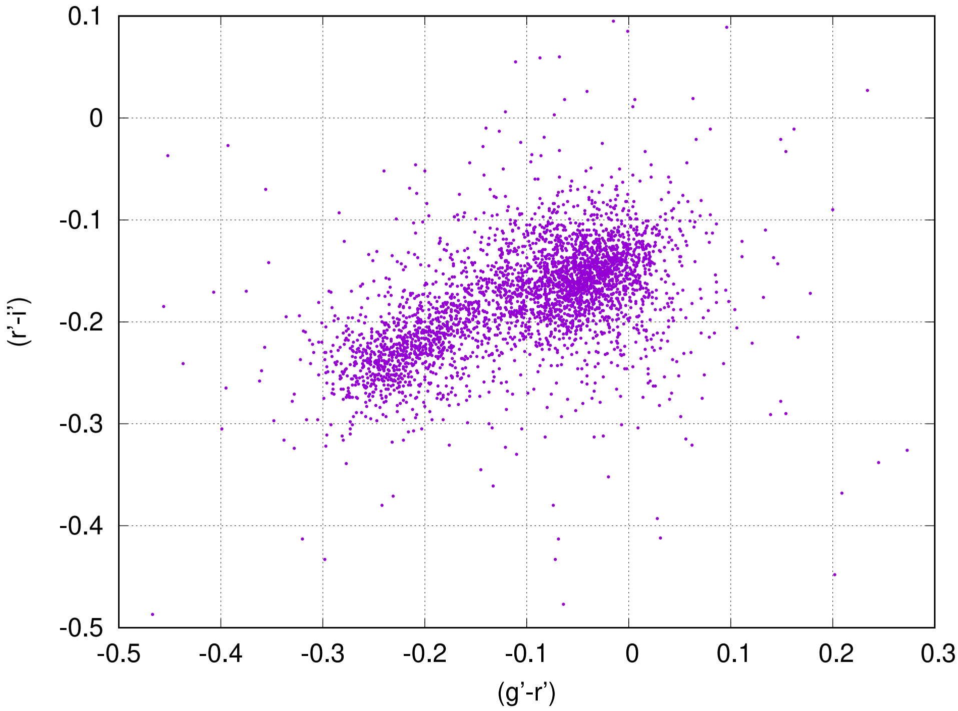

Figure 1: $(r' - i')$ on $(g' - r')$: the most populated area

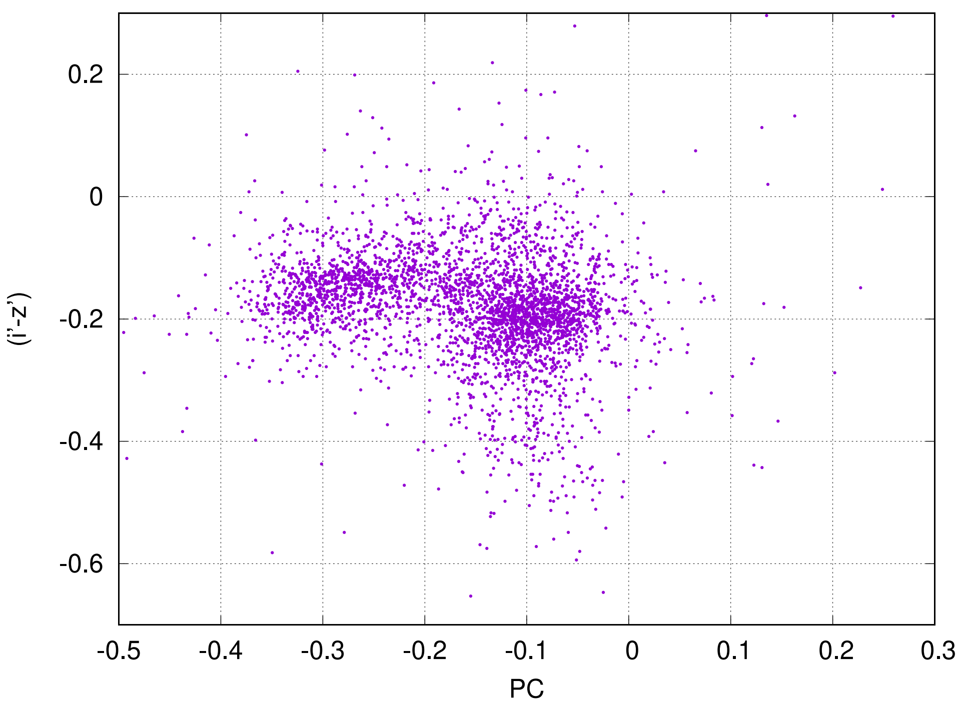

Figure 2: $(i' - z')$ on PC: the most populated area

Marking Scheme:

- Question (a)
  1. $(g' - r';\ r' - i')$ plot — **2 pt.**
  2. $n = 3504$ — **3 pt.**
     OR $n \in [3500;\ 3510]$ — 2 pt., $n \in [3300;\ 3700]$ — 1 pt.
- Question (b)
  3. $a = 0.40$ — **5 pt.**
     For every deviation step of 0.01, 1 point is deducted, e. g. the answer $a = 0.37$ is worth 2 pt.
     The score for this question cannot be negative.

4. Question (c): (PC; $i' - z'$) plot — **3 pt.**
5. Question (d): spectral type with justification — **2 pt.**
6. Question (e): correct number for each spectral type — **1 pt**. × **5** = **5 pt**.

2 Support Service

An analemma (from Ancient Greek αναλημμα – "support") is a diagram showing the position of the Sun in the sky as seen from a fixed location on the Earth at the same mean solar time, as that position varies over the course of a year. It is directly related to the so-called equation of time — the difference between mean and true solar time (that is, the difference between the right ascensions of the true and mean Sun):

$$\eta = t_{\text{mean}} - t_{\text{true}} = \alpha_{\text{true}} - \alpha_{\text{mean}}.$$

This difference is due to two reasons: the tilt of the planet's axis of rotation (obliquity) and the eccentricity of its orbit.

Assume the planet pass perihelion at the moment $t = 0$. If the planet moved uniformly, its position would be determined by the mean anomaly

$$M(t) = \frac{2\pi t}{T},$$

where T is the orbital period of the planet.

The actual position of the planet in its orbit is determined by the true anomaly ν. The true anomaly ν is geometrically related to the eccentric anomaly E, which shows the position of some imaginary point on the auxiliary circle (see Figure 3).

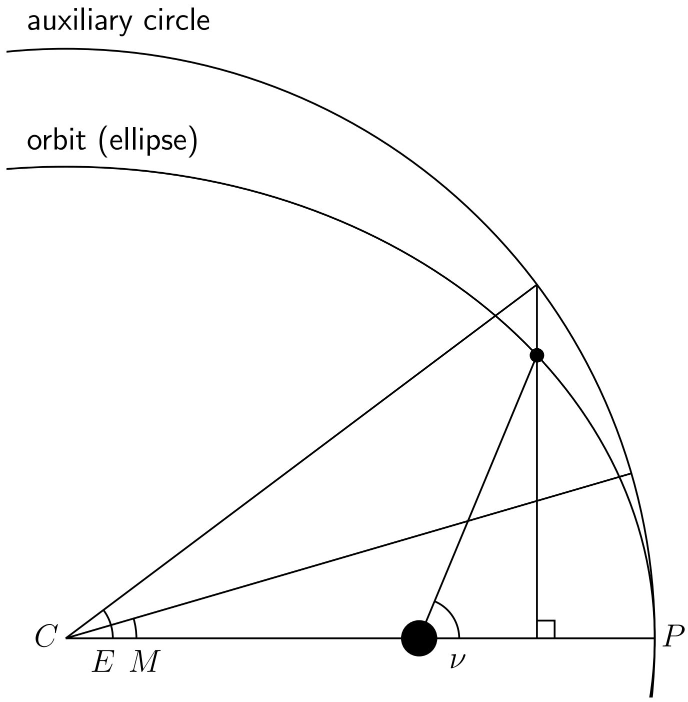

Figure 3: Geometric meaning and connection between different anomalies. C is the center of the ellipse and auxiliary circle, P is the pericenter of the orbit

Please note that

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2},$$

where e is the eccentricity of the planet's orbit. Also, the mean and eccentric anomalies (expressed in radians) are related by Kepler's equation:

$$E - e \cdot \sin E = M.$$

a) Plot the difference $(\nu - M)$ as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars. For convenience, express $(\nu - M)$ in "local" minutes: 1 solar day = 24 hours, 1 hour = 60 minutes.

Let x be the ecliptic longitude of the Sun measured from the vernal equinox for the planet when this planet is at perihelion. You can find the corresponding values in the Constants Table.

b) Express the "local" ecliptic longitude of the Sun λ in terms of ν and x.

Hour angles and right ascensions are measured along the celestial equator. Since the ecliptic is inclined to the celestial equator by an angle of ε, both of these parameters are related to the ecliptic longitude nonlinearly. For example, for right ascension α we have:

$$\tan\alpha = \tan\lambda \cdot \cos\varepsilon.$$

c) Plot the difference $(\alpha - \lambda)$ expressed in "local" minutes as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars.

The equation of time can be represented as the sum

$$\eta(t) = (\nu - M) + (\alpha - \lambda).$$

d) Plot the equation of time η expressed in "local" minutes as a function of time $t \in [0;\ T]$ for the orbits of the Earth and Mars.

The declination of the Sun δ can be expressed by the formula

$$\sin\delta = \sin\lambda \cdot \sin\varepsilon.$$

e) Plot the analemma corresponding to mean noon for an observer at mid-latitudes of the Northern Hemisphere of the Earth and Mars (north is up, west is to the right).

f) There are the analemmas of five imaginary planets in Figure 4. Determine the parameters e, x, and ε for each model. In case there are several possible answers, please provide any.

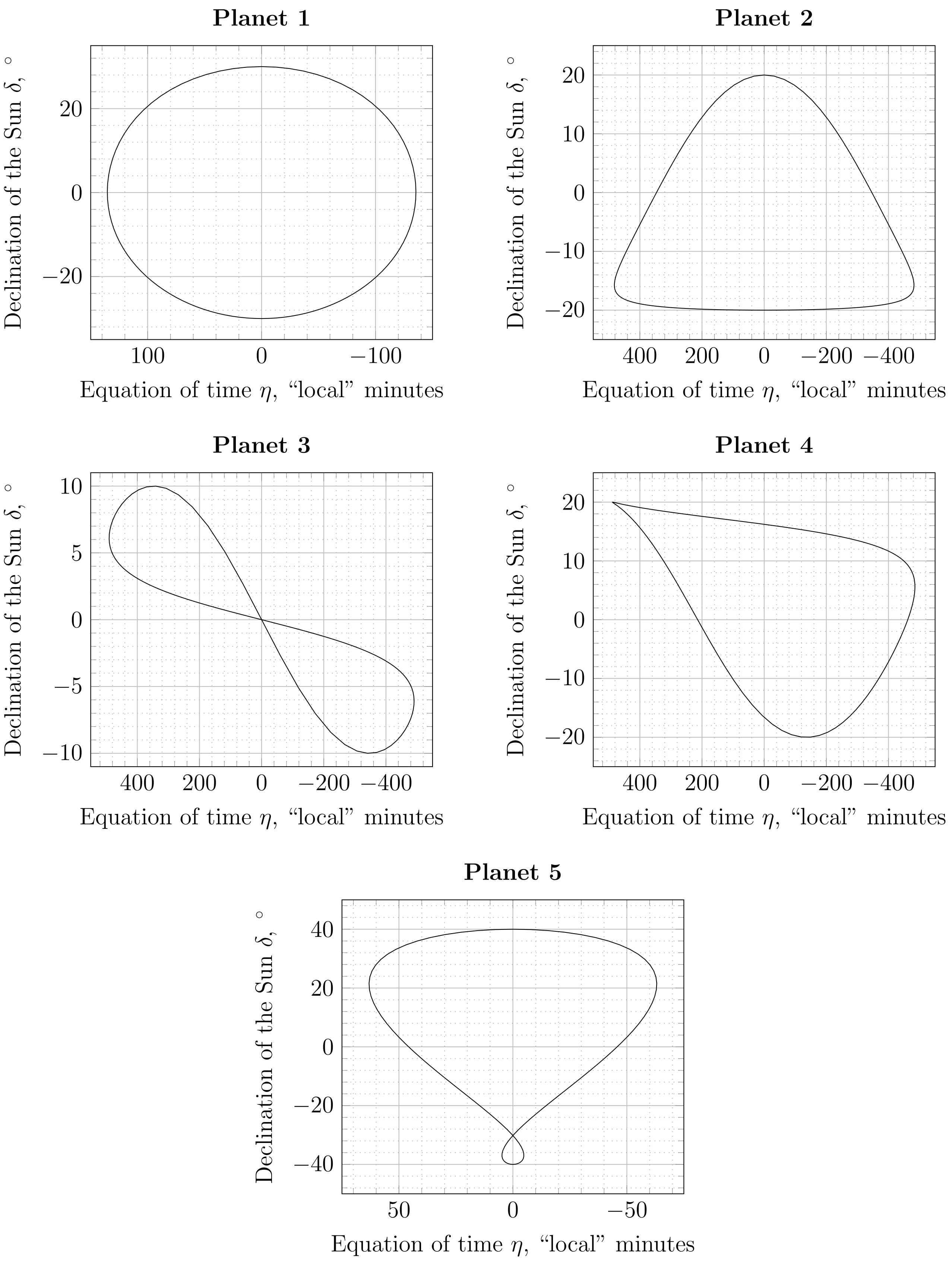

Figure 4: Analemmas of five imaginary planets

Solution:

a) It is convenient to use the eccentric anomaly E as a parameter. For $E = 0$ to 2π we can tabulate calculations in radians:

$$M = E - e \cdot \sin E, \qquad t = \frac{M}{2\pi} \cdot T;$$
$$\nu = 2\arctan\left(\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2}\right) + 2\pi k,$$

where $k = 0$ if $E \leqslant \pi$, and $k = 1$ if $E > \pi$.

To convert the obtained results from radians to "local" minutes we have to multiply them by factor $\frac{24\times60}{2\pi} = \frac{720}{\pi}$.

b) Parameters λ, ν, and x are measured along the "local" ecliptic in the same direction. Also $\Delta\lambda = \Delta\nu$. When the planet is at the perihelion, $\lambda = x$ and $\nu = 0$, so

$$\lambda = \nu + x.$$

c) The right ascension

$$\alpha = \arctan(\tan\lambda \cdot \cos\varepsilon) + \pi n,$$

where integer n is chosen so that $0 \leqslant \alpha < \pi$ when $0 \leqslant \lambda < \pi$, and $\pi \leqslant \alpha < 2\pi$ when $\pi \leqslant \lambda < 2\pi$. Please do not forget to convert the results to the local minutes!

d–e) We have everything we need to plot the following graph:

$$\begin{cases} \eta = (\nu - M) + (\alpha - \lambda); \\ \delta = \arcsin(\sin\lambda \cdot \sin\varepsilon). \end{cases}$$

See Figures 5 and 6.

f) Varying parameters e, x, and ε, we can adjust the calculated analemmas to the model ones.

Note that the maximum Sun declination for each planet is ε. A symmetric figure means that x is divisible by $\pi/2 = 90°$.

The possible values of e, x, and ε for each model planet are shown in the table below:

Planet	1	2	3	4	5
e	0.3	0.9	0.9	0.9	0.1
x, °	90	90	0	305	90
ε, °	30	20	10	20	40

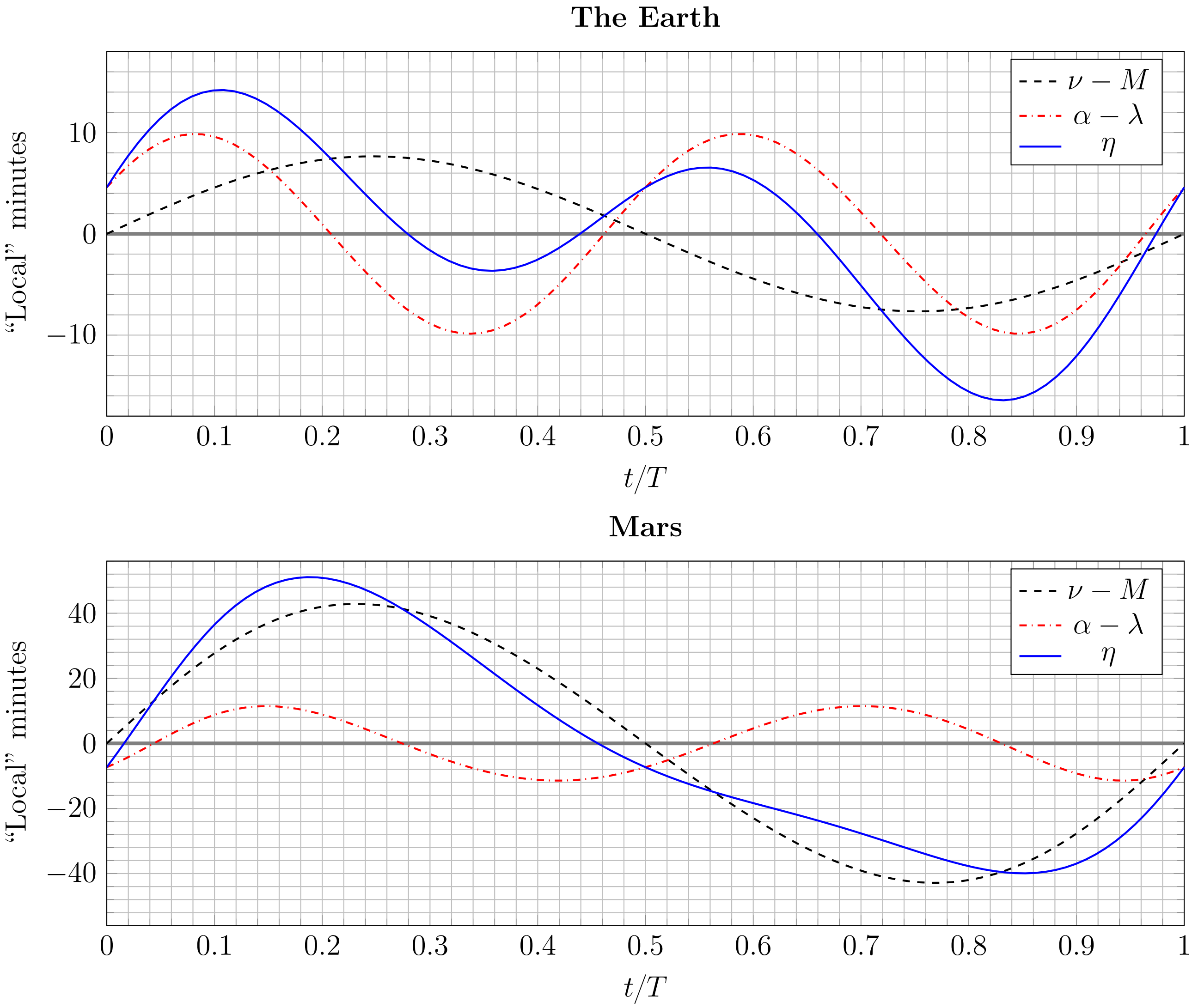

Figure 5: Composition of the equation of time for the Earth and Mars

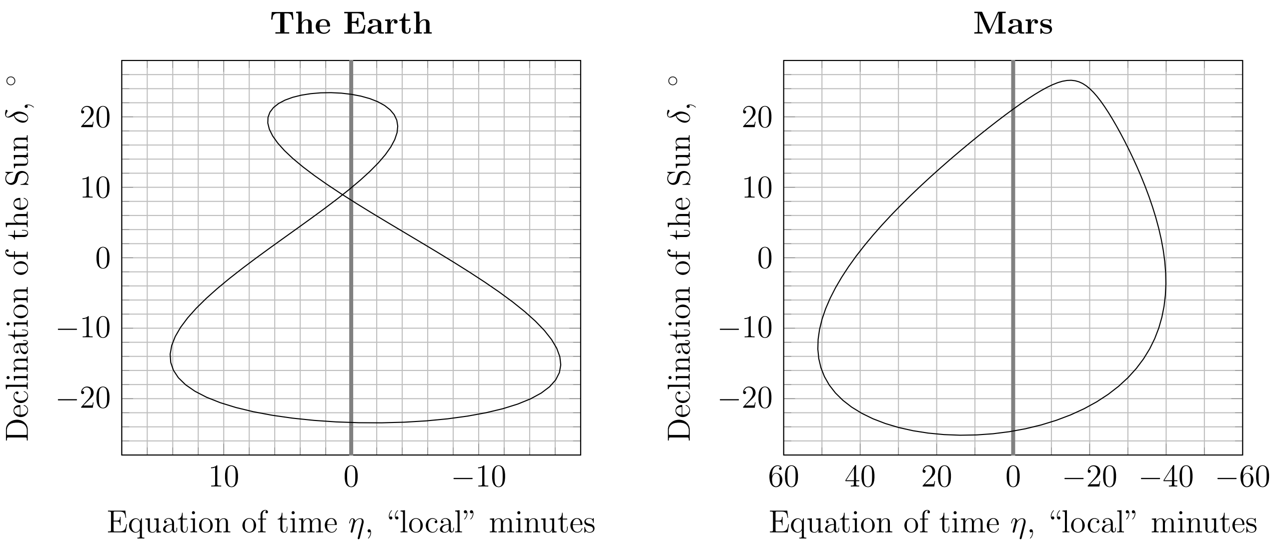

Figure 6: Calculated analemmas for the Earth and Mars

Marking Scheme:

1. Question (a) — **3 pt.**
2. Question (b) — **1 pt.**
3. Question (c) — **3 pt.**
4. Question (d) — **2 pt.**
5. Question (e) — **4 pt.**
6. Question (f)
   - ε: method — **1 pt.** + values — **0.4 pt**. × **5** = **2 pt.**
   - x: method — **1 pt.** + values — **0.2 pt.** × **5** = **1 pt.**
   - e: method — **1 pt.** + values — **0.2 pt.** × **5** = **1 pt.**

Note. In the original formulation of the problem, the notation x could be understood ambiguously. Using the well-known ideas about the Earth's orbit (the Earth passes the perihelion on January 2–5) one could easily guess how to interpret this designation correctly. However, solutions with an "inverted" ecliptic longitude of the pericenter are assessed without penalty.

3 Ephemeric Quest

The file ephem.dat contains ephemerides of some small body of the Solar system. The first and second columns of the file contain the date and universal time of the observation, followed by the right ascension (in hours, minutes and seconds) and declination (in degrees, minutes and seconds) for the observer on the Earth (geocentric location):

Date	Time	Right ascension	Declination
2023-Mar-18	00:00:00	$23^{\mathrm{h}}\,59^{\mathrm{m}}\,03.88^{\mathrm{s}}$	$0^\circ\,44'\,46.9''$
2023-Mar-19	00:00:00	$0^{\mathrm{h}}\,0^{\mathrm{m}}\,41.94^{\mathrm{s}}$	$0^\circ\,55'\,34.0''$
...	...	...	...

The absolute magnitude of the object is $H = 18.51^{\mathrm{m}}$. The orbit of the object is circular, the inclination is small.

a) Determine whether the object's orbit lies inside or outside the Earth's orbit.

b) Estimate the radius of the object's orbit.

c) Estimate the minimal time interval between two consecutive greatest elongations or consecutive quadratures.

d) Determine the difference between the maximal and minimal apparent magnitudes of the object for an observer on the Earth during one synodic period.

e) Plot the dependence of the apparent magnitude of the object on time for an observer on the Earth during one synodic period.

Hint: for the Solar system objects, the absolute magnitude, commonly called H, is defined as the apparent magnitude that the object would have if it were one astronomical unit (1.00 au) from both the Sun and the observer, and in conditions of ideal solar opposition (the apparent phase is 1.00).

Solution:

a) First of all, we plot the positions of an object in the sky in equatorial coordinates. Note that the right ascensions of the object vary widely. Let us look at the area of the graph that is mostly interesting for us, where we can see a loop of apparent retrograde motion (see Figure 7). The color on the graph shows the number of the observation day, for convenience.

The center of the retrograde motion arc corresponds to the observations in the winter of 2023. But the right ascensions of the object correspond to the summer zodiac constellations. Consequently, the retrograde motion arc is observed in the vicinity of the opposition, and the object is superior to the Earth.

b) There are several ways to determine the radius of an object's orbit.

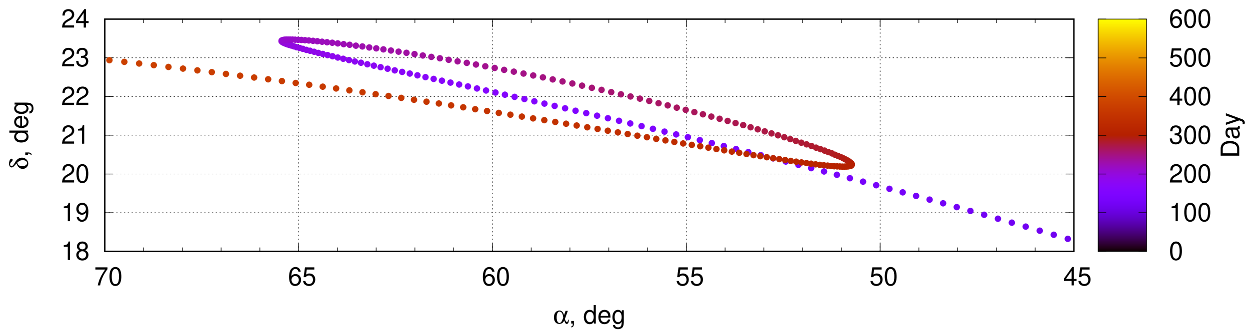

Figure 7: The positions of an object on the celestial sphere observed from the Earth. The equatorial coordinates in degrees are plotted along the axes. The color shows the number of the day since the start of the observation

Method 1. Let us analyze the simplest one. We can determine the point of opposition of the object and estimate the angular velocity of its movement across the sky.

Let a be the radius of the object's orbit, $a_\oplus$ be the radius of the Earth's orbit, V be the orbital velocity of the object, $V_\oplus$ be the velocity of the Earth. The angular velocity of an object moving across the sky in its opposition is

$$\omega = \frac{V_\oplus - V}{a - a_\oplus}.$$

Since the object's orbit is circular, the velocity is inversely proportional to the square root of the radius of the orbit:

$$\omega = \frac{V_\oplus - V_\oplus \cdot \sqrt{\dfrac{a_\oplus}{a}}}{a - a_\oplus} = \frac{V_\oplus}{a} \cdot \frac{1 - \sqrt{\dfrac{a_\oplus}{a}}}{1 - \dfrac{a_\oplus}{a}} = \frac{V_\oplus}{a} \cdot \frac{1}{1 + \sqrt{\dfrac{a_\oplus}{a}}} = \frac{V_\oplus}{a + \sqrt{a a_\oplus}}.$$

Now we have the equation for the radius of the object's orbit:

$$a + \sqrt{a a_\oplus} - \frac{V_\oplus}{\omega} = 0.$$

The opposition occurs approximately on November 21. Let us determine how far the object has moved in 4 days in the vicinity of this date. One can take a different time interval, so the answers may be slightly different

19.11.2023:	$\alpha_1 = 3^\mathrm{h}\,55^\mathrm{m}\,47.92^\mathrm{s}, \quad \delta_1 = 22^\circ\,32'\,37.1'',$
23.11.2023:	$\alpha_2 = 3^\mathrm{h}\,51^\mathrm{m}\,45.64^\mathrm{s}, \quad \delta_2 = 22^\circ\,20'\,06.2''.$

The angle θ traversed by the object is estimated by Pythagorean theorem taking into account the correction for small circles: $\theta \approx \sqrt{(\delta_1 - \delta_2)^2 + (\alpha_1 - \alpha_2)^2 \cos^2\delta_1} \approx 0.956^\circ$, or by the spherical law of cosines: $\theta = \arccos\left(\sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2)\right) \approx 0.956^\circ$.

So an estimation of the angular velocity is

$$\omega = \frac{0.956^\circ}{4\ \mathrm{d}} = 0.239^\circ/\mathrm{d}.$$

Substituting this value into the equation above, we obtain

$$\frac{V_\oplus}{\omega} = \frac{V_\oplus}{\omega_\oplus}\cdot\frac{\omega_\oplus}{\omega} = a_\oplus\cdot\frac{360^\circ}{\omega T_\oplus} = a_\oplus \times \frac{360^\circ}{0.239^\circ/d \times 365.26^{\mathrm{d}}} = 4.124 a_\oplus$$

$$\Longrightarrow \quad \frac{a}{a_\oplus} + \sqrt{\frac{a}{a_\oplus}} = 4.124.$$

This equation can be solved using iterative method:

$$\frac{a_{n+1}}{a_\oplus} = 4.124 - \sqrt{\frac{a_n}{a_\oplus}}, \qquad a_0 = 1\ \mathrm{au},$$

that quickly converges to $a = 2.57$ au or $3.8\cdot 10^{11}$ m. The data table corresponds to the asteroid 2021 GK_{139}. The real value for the semi-major axis of its orbit is 2.595 au.

Method 2. A more complex method requires the use of the information about the stationary points. Let us determine the relation between the radius of the object's orbit and the length of the arc of its retrograde motion.

We introduce a coordinate system centered on the Earth as shown in Figure 8. The coordinates of the object in such a system are $(a\sin\lambda;\ a\cos\lambda - a_\oplus)$. The relative velocity of the object is $(V\cos\lambda - V_\oplus,\ -V\sin\lambda)$.

At the stationary points, the observed relative velocity of the object is directed along the line of sight. We can write the proportionality

$$\frac{V\cos\lambda - V_\oplus}{a\sin\lambda} = \frac{-V\sin\lambda}{a\cos\lambda - a_\oplus} \quad\Longrightarrow\quad \frac{\sqrt{a_\oplus/a}\cdot\cos\lambda - 1}{a\sin\lambda} = \frac{-\sqrt{a_\oplus/a}\cdot\sin\lambda}{a\cos\lambda - a_\oplus}.$$

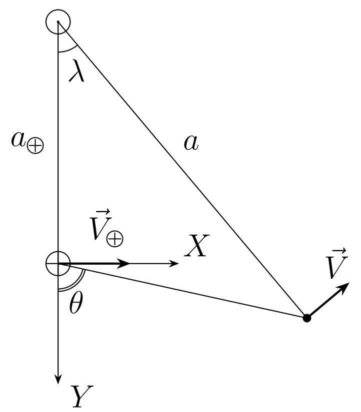

Figure 8: Determining the positions of the stationary points

If we measure a in AU, we'll get

$$\sqrt{a}\cos^2\lambda - \frac{1}{\sqrt{a}}\cos\lambda - a\cos\lambda + 1 = -\sqrt{a}\sin^2\lambda \quad\Longrightarrow\quad \cos\lambda = \frac{\sqrt{a}+1}{a+1/\sqrt{a}}.$$

Based on the track of an object on the celestial sphere, we determine the length of the retrograde motion arc. The distance between the stationary points turns out to be about 14°. This value can be measured directly from the plot, taking into account the projection effect, or estimated using the spherical law of cosines based on the coordinates of the stationary points. The stationary points themselves correspond approximately to October 6, 2023 and January 9, 2024. During this time, due to the movement of the Earth in its orbit, the observer's line of sight will rotate 95°, so the corresponding angle θ in Figure 8 equals to $(95° + 14°)/2 \approx 54°$.

By the law of sines

$$\frac{\sin\theta}{a} = \frac{\sin(\theta-\lambda)}{a_\oplus}.$$

We need to find both the radius a and the angle λ. It seems meaningful to determine the roots numerically. For example, we use the bisection-like method on the segment $[1.5; 10]$ au and calculate λ from both the sine theorem (λ_1) and the $\cos\lambda$ estimate (λ_2):

a	1.5	10.0	5.0	3.0	2.5
λ_1, °	16.2	66.2	53.6	40.2	34.5
λ_2, °	21.4°	49.4	44.7	38.4	35.1

We can take the value of 2.5 au as a fair estimation.

Method 3 (sketch). Finally, one can simply convert the object's coordinates to the ecliptic system, calculate the ecliptic longitude of the Sun (for details see Problem 2) and the object's elongation at each point. Next, we will notice two points, for example, with zero elongation (the conjunction of the object with the Sun). The time difference at these points corresponds to the synodic period of the object. Then it is easy to determine the sidereal period, and then the radius of the object's orbit.

c) It was previously shown that the object is superior. Then the concept of the greatest elongation is meaningless for it.

Since the time interval between quadratures is related to the value of the synodic period, let us first estimate the synodic period using Kepler's third law:

$$S = \frac{T\cdot T_\oplus}{T-T_\oplus} = \frac{a^{1.5}\cdot a_\oplus^{1.5}}{a^{1.5}-a_\oplus^{1.5}} = 1.3\ \text{yr}.$$

We should estimate the angle θ centered on the Sun between the direction to Earth and the object in quadrature (see Figure 9): $\theta = \arccos(a_\oplus/a) \approx 67°$.

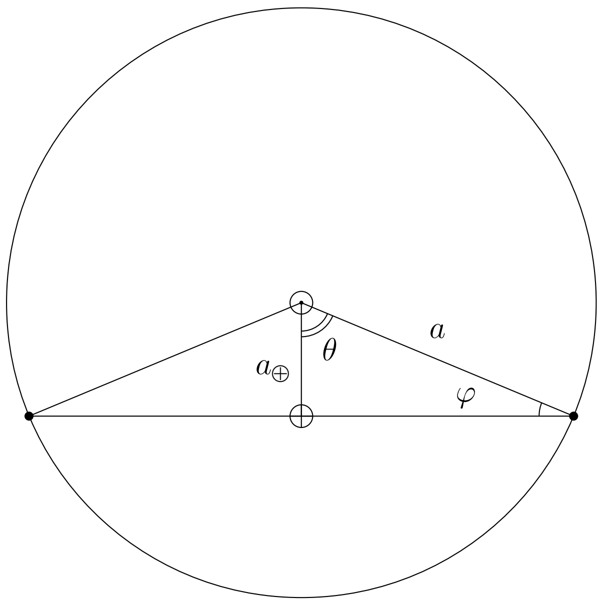

Figure 9: Geometry of the quadratures of the object

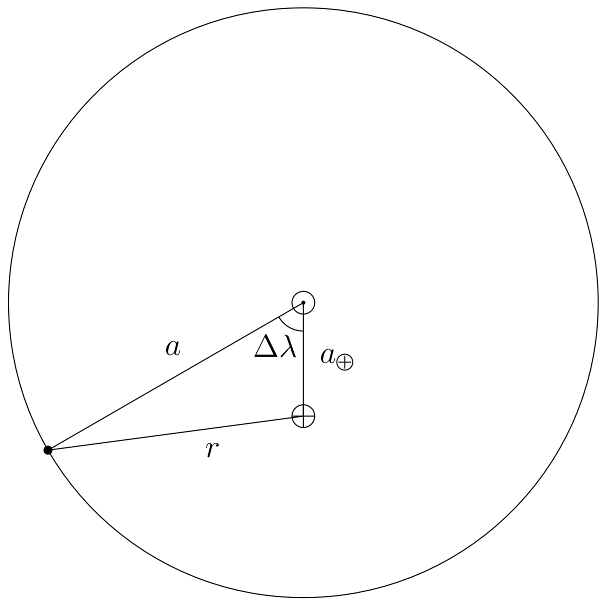

Figure 10: Geometry for an arbitrary configuration of the object

Then the minimal time interval between successive quadratures is $\Delta T = \dfrac{2\theta}{360^\circ} \cdot S = 0.49$ yr.

d) Let us think about the moments when the minimal and maximal magnitudes are reached. Note that the magnitude dependence on the observed phase is insignificant. The minimal phase corresponds to the point of quadrature,

$$\Phi_{\min} = \frac{1+\cos\varphi}{2} = \frac{1+\cos\arcsin(a_\oplus/a)}{2} \approx 0.96.$$

This value differs very slightly from 1, and within the accuracy of our calculations, we can ignore the phase change.

The object's orbit is circular, so the most significant factor is the difference in the distances between the Earth and the object. The object becomes the faintest in its conjunction, and the brightest in its opposition:

$$m_o - m_c = -2.5\lg\frac{E_o}{E_c} = -2.5\lg\frac{(a-a_\oplus)^{-2}}{(a+a_\oplus)^{-2}} = 5\lg\frac{a-a_\oplus}{a+a_\oplus} = -1.8^{\mathrm{m}}.$$

e) We have to express the dependence of the apparent magnitude on the distance r to the object. The illuminance from the Sun is inversely proportional to the square of the distance from the Sun to the object. Also, the illuminance created by an object on the Earth is inversely proportional to the square of the distance from the object to the Earth. We use Pogson's law and write

$$m(r) - H = -2.5\lg\frac{a_\oplus^2 a_\oplus^2}{r^2 a^2} = 5\lg\frac{ra}{a_\oplus^2}\,.$$

If we express distances in astronomical units, the formula takes the form

$$m(r) = H + 5\lg ra.$$

Let $t = 0$ correspond to the moment of the opposition. At the time t, the difference between the heliocentric longitudes of the Earth and the object is

$$\Delta\lambda = \frac{t}{S} \cdot 360^\circ.$$

The distance to the object is related to the $\Delta\lambda$ with the law of cosines (see Figure 10):

$$r = \sqrt{a_\oplus^2 + a^2 - 2a_\oplus a \cos\Delta\lambda}.$$

In au, $r = \sqrt{1 + a^2 - 2a\cos\Delta\lambda}$. Finally, we obtain the dependence of the apparent magnitude on the number of days since the opposition

$$\begin{aligned} m(t) &= H + 5\lg(a\sqrt{1 + a^2 - 2a\cos\Delta\lambda}) = \\ &= H + 5\lg a + 2.5\lg\left(1 + a^2 - 2a\cdot\cos\left(\frac{t}{S}\cdot 360^\circ\right)\right). \end{aligned}$$

The graph of this dependence is shown in Figure 11.

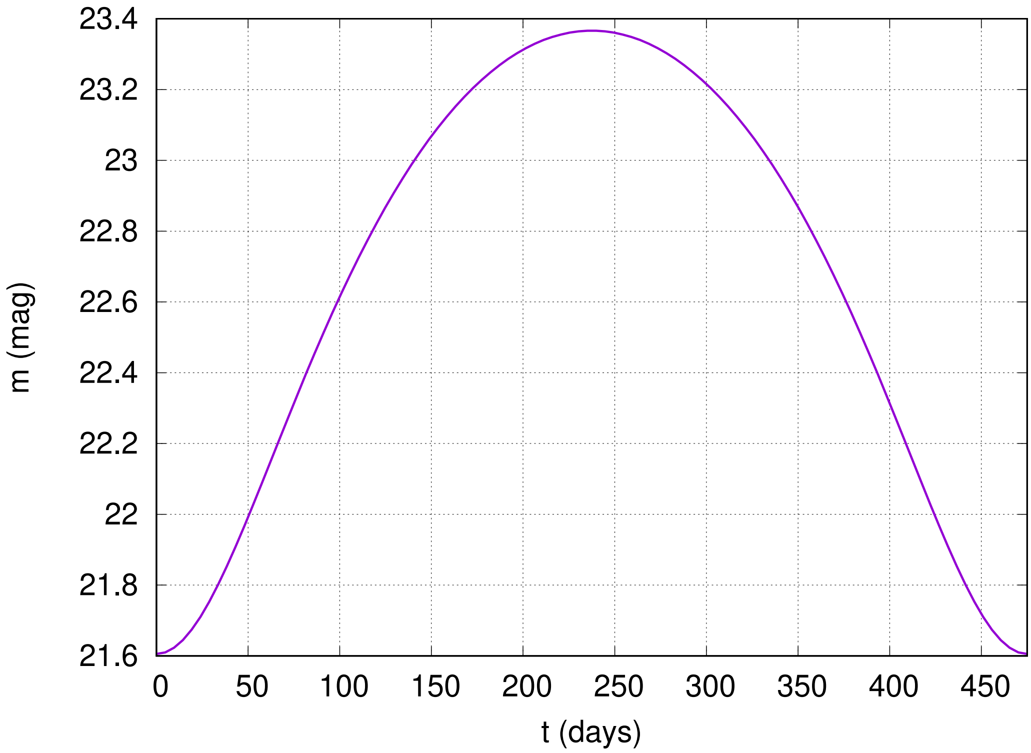

Figure 11: The object's apparent magnitude on the number of days since the opposition

Marking Scheme:

- Question (a)
  1. Description of the method for determining the location of the object's orbit — **2 pt.**
  2. The answer — **1 pt** *is given only if the description of the method is provided.*
- Question (b)
  3. Description of the method for determining the radius of the orbit, taking data from the graph, data processing — **2 pt.** + **1 pt.** + **1 pt.**
  4. The answer — **1 pt** *is given only if the description and data processing are provided.*
- Question (c)
  5. Understanding possible configurations, choosing the necessary data — **1 pt.**
  6. Writing the correct formulae and the final expression — **1 pt.** + **1 pt.**
  7. Answer — **1 pt**.
- Question (d)
  8. Understanding the necessary configurations, choosing the correct data — **1 pt.**
  9. Writing the correct formulae and the final expression — **1 pt.** + **1 pt.**
  10. Answer — **1 pt**.
- Question (e)
  11. Dependence of the apparent magnitude on the date — **2 pt.**
  12. Calculation of values, plot — **1 pt.** + **1 pt.**

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R} = 8.314$ J/(mol · K)
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au $= 149.6 \cdot 10^6$ km
Parsec	1 pc $= 206\,265$ au
Hubble constant	$H_0 = 70$ (km/s)/Mpc

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Mass	$\mathfrak{M}_\odot = 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot = 4.74^\mathrm{m}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Earth

Radius	$R_\oplus = 6371$ km
Sidereal day	$\tau_\oplus = 23^\mathrm{h}\,56^\mathrm{m}\,04^\mathrm{s}$
Obliquity of ecliptic	$\varepsilon_\oplus = 23.437°$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^\mathrm{d}$
Orbital eccentricity	$e_\oplus = 0.0167$
	$x_\oplus = 283.3°$

Mars

Radius	$R_M = 3390$ km
Sidereal day	$\tau_M = 24^\mathrm{h}\,37^\mathrm{m}\,23^\mathrm{s}$
Obliquity (axial tilt)	$\varepsilon_M = 25.192°$
Orbital semi-major axis	$a_M = 1.52$ au
Orbital period	$T_M = 1.88$ yr
Orbital eccentricity	$e_M = 0.0934$
	$x_M = 251°$