1 Crossroad of Worlds

The table below shows the current equatorial coordinates (α, δ), parallax ϖ, proper motion components (μ_α, μ_δ), and radial velocity V_r for two stars for the observer on the Earth. Consider the motion of the stars relative to the Sun to be linear. Determine when the angular distance between the stars will reduce by 50 % as observed from the Earth.

Star	(α, δ)	ϖ (mas)	μ_α (mas/yr)	μ_δ (mas/yr)	V_r (km/s)
A	$(10^{\mathrm{h}}\,48^{\mathrm{m}}\,00^{s},\ 0^\circ\,0'\,0'')$	80	-160	0	-25
B	$(10^{\mathrm{h}}\,00^{\mathrm{m}}\,00^{\mathrm{s}},\ 0^\circ\,0'\,0'')$	60	$+180$	0	-20

Solution:

The stars are in the plane of the celestial equator. Since there are no components of proper motion in declination, the movement of stars also occurs in the plane of the celestial equator, $\mu = |\mu_\alpha|$.

Let us determine the distances to each of the stars and their tangential velocities:

$$r\,[\mathrm{pc}] = \frac{1}{\varpi\,['']},$$

$$V_\tau\,[\mathrm{km/s}] = 4.74 \cdot \mu\,[''/\mathrm{yr}] \cdot r\,[\mathrm{pc}] = 4.74 \cdot \frac{\mu\,[\mathrm{mas/yr}]}{\varpi\,[\mathrm{mas}]};$$

$$V_{\tau,A} = 4.74 \times \frac{160}{80} = 9.5\ \mathrm{km/s},$$

$$V_{\tau,B} = 4.74 \times \frac{180}{60} = 14.2\ \mathrm{km/s}.$$

We also calculate the overall velocity of each star relative to the Sun:

$$V_A = \sqrt{V_{\tau,A}^2 + V_{r,A}^2} = 26.7\ \mathrm{km/s},$$

$$V_B = \sqrt{V_{\tau,B}^2 + V_{r,B}^2} = 24.5\ \mathrm{km/s}.$$

Currently, the angular distance between the stars is

$$\theta_0 = 48^{\mathrm{m}} = 12^\circ.$$

We need to determine when this distance will decrease to $0.5\theta_0 = 12^\circ \times 0.5 = 6.0^\circ$.

Since the radial velocities of the stars are negative, the stars are currently moving towards the Sun. The signs of the components of their proper motion in right ascension differ, so the stars really will come “closer” in the Earth’s sky.

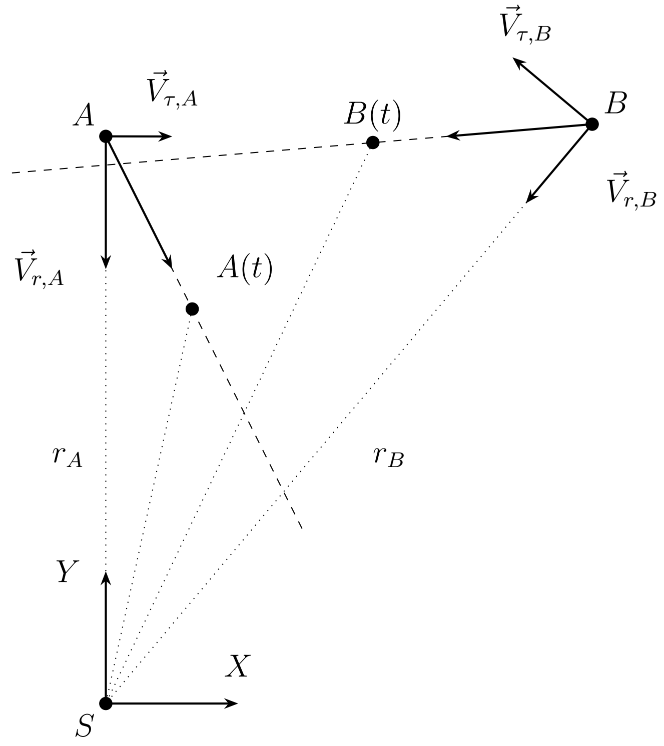


Figure 1: The motion of the stars relative to the Sun in the equatorial plane (not to scale)

Method 1. Let us introduce a coordinate system centered on the Sun as shown in Figure 1. After a time t from the current moment, the coordinates of the star A will be

$$X_A(t) = V_{\tau,A} \cdot t,$$
$$Y_A(t) = r_A - V_{r,A} \cdot t\,.$$

The angle between the full velocity vector of the star B and its tangential velocity is $\arctan \frac{|V_{\tau,B}|}{|V_{r,B}|}$, therefore the angle of the velocity vector of the star B to the X-axis is

$$\varphi = -90^\circ - \theta_0 - \arctan \frac{|V_{\tau,B}|}{|V_{r,B}|} = -137.4^\circ.$$

After a time t from the current moment, the coordinates of the star B will be

$$X_B(t) = r_B \sin\theta_0 + V_B \cos\varphi \cdot t,$$
$$Y_B(t) = r_B \cos\theta_0 + V_B \sin\varphi \cdot t.$$

The angle $\theta(t)$ between the directions to the stars is calculated as the angle between the vectors $\mathbf{SA}(t)$ and $\mathbf{SB}(t)$:

$$\cos\theta(t) = \frac{\mathbf{SA}(t)\cdot\mathbf{SB}(t)}{|\mathbf{SA}(t)|\cdot|\mathbf{SB}(t)|} = \frac{X_A(t)X_B(t) + Y_A(t)Y_B(t)}{\sqrt{X_A(t)^2 + Y_A(t)^2}\cdot\sqrt{X_B(t)^2 + Y_B(t)^2}}\,.$$

We will measure distances in parsecs, time in thousands of years and velocities in parsec per thousand years. Next, we substitute the numerical data into the expressions for the coordinates:

$$1\ \text{km/s} = \frac{3.16 \cdot 10^7\ \text{s/yr}}{206\,265 \times 149.6 \cdot 10^6\ \text{km/pc}} = 1.02 \cdot 10^{-6}\ \text{pc/yr} = 1.02 \cdot 10^{-3}\ \text{pc/kyr};$$

$$\begin{cases} X_A(t) = 0.0097t, \\ Y_A(t) = 12.5 - 0.026t; \end{cases} \qquad \begin{cases} X_B(t) = 3.47 - 0.018t, \\ Y_B(t) = 16.3 - 0.017t. \end{cases}$$

So our goal is to find the roots of the function

$$f(t) = \cos 6^\circ - \frac{0.0097t \cdot (3.47 - 0.018t) + (12.5 - 0.026t) \cdot (16.3 - 0.017t)}{\sqrt{(0.0097t)^2 + (12.5 - 0.026t)^2} \cdot \sqrt{(3.47 - 0.018t)^2 + (16.3 - 0.017t)^2}} .$$

The equation is similar to the 4th degree equation. We will solve it numerically.

For geometric reasons, the equation must have exactly 2 roots. At some point, the stars will be on the same line of sight for the observer. Therefore, an angular distance of 6 degrees will occur both before and after such a moment.

First, let us define the ranges of the argument t at which the function $f(t)$ changes its sign. To do this, we calculate the values of the function on some grid:

t [kyr]	0	20	40	60	80	100	120	140	160
$f(t) \cdot 10^3$	16.4	10.0	4.4	−0.28	−3.6	−5.3	−5.0	−2.0	4.4
			+	−				−	+

We can find roots on the intervals [40 : 60] and [140 : 160] using the bisection method:

t [kyr]	40	60	50	55	58	59
$f(t) \cdot 10^3$	4.4	−0.28	1.9	0.77	0.13	−0.08
t [kyr]	140	160	150	145	147	148
$f(t) \cdot 10^3$	−2.0	4.4	0.75	−0.71	−0.16	0.14

As a result, the angular distance of 6° is reached after 59 thousand years and 148 thousand years from the current moment. Due to emerging rounding errors, the answers may differ from those indicated above by several thousand years.

Method 2. The geometrical framework here is the same as used in Method 1, but the calculations are much simpler.

Firstly, we adopt the local coordinate system for each star $i \in A, B$ at the current moment. We denote the first coordinate x_i being along the line of sight at the initial moment of time $t = 0$. Second coordinate in the equatorial plane y_i is measured along the perpendicular to the x_A direction and is positive in the direction counterclockwise from polar axis.

Then the initial coordinates for each star i are:

$$x_i(t = 0) = r_i\,,$$
$$y_i(t = 0) = 0.$$

Velocities measured along those axes are the radial and the tangential one respectively.

After a time t passes the coordinates should be

$$x_i(t = 0) = r_i + V_{r,i} \cdot t,$$
$$y_i(t = 0) = V_{\tau,i} \cdot t$$

Calculating all the velocities and distances we get the coordinates (in parsec) expressed as function of time t (in million years):

$$\begin{cases} x_A(t) = 16.67 - 1.023 \cdot 25 \cdot t, \\ y_A(t) = 1.023 \cdot 9.48 \cdot t; \end{cases} \qquad \begin{cases} x_B(t) = 12.50 - 1.023 \cdot 20 \cdot t, \\ y_B(t) = 1.023 \cdot 14.22 \cdot t. \end{cases}$$

Coordinate frame of star A is rotated relatively the coordinate frame of star B through the angle θ_0 counterclockwise. To transform the coordinates into the same coordinate system we perform the rotation of coordinates $(x_B, y_B) \to (x'_B, y'_B)$:

$$\begin{cases} x'_B = x_B \cos\theta_0 - y_B \sin\theta_0\,, \\ y'_B = x_B \sin\theta_0 + y_B \cos\theta_0\,. \end{cases}$$

After that we should express the angle θ between A and B as a function of time. Using the given cartesian coordinates we can write a dot product of the position vectors:

$$\mathbf{r}_A \cdot \mathbf{r}_B = r_A r_B \cos\theta(t) = x'_B x_A + y'_B y_A \quad \Longrightarrow \quad \theta(t) = \frac{x'_B x_A + y'_B y_A}{r_A r_B};$$

$$\cos 6^\circ =$$
$$= \frac{(16.67 - 1.023 \cdot 25 \cdot t) \cdot x'_B(t) + (1.023 \cdot 9.48 \cdot t) \cdot y'_B(t)}{\sqrt{\left[(16.67 - 1.023 \cdot 25 \cdot t)^2 + (1.023 \cdot 9.48 \cdot t)^2\right]\left[(12.50 - 1.023 \cdot 20 \cdot t)^2 + (1.023 \cdot 14.22 \cdot t)^2\right]}}\,.$$

The answers can be found using bisection method. Note that we have 2 answers, any of them will give participant a full score, since the angle of 6° between star can occur before and after the alignment of such stars on the sky.

Such answers are $t_1 = 0.055$, $t_2 = 0.141$ in millions of years.

The results are slightly different because the calculations in this problem are sensitive to rounding.

Method 3. Using the same systems of coordinates as were in Method 2, one can say that the angle star travelled on the sky up to time t from initial moment corresponding to parameters given in the table is

$$\theta_i = \arctan\left(\frac{|V_{\tau,i}|t}{r_i - |V_{r,i}|t}\right).$$

From the signs of proper motions and radial velocities it is obvious that the stars are moving towards each other. Then the equation to solve for the first moment of time is

$$\arctan\left(\frac{|V_{\tau,A}|t}{r_A - |V_{r,A}|t}\right) + \arctan\left(\frac{|V_{\tau,B}|t}{r_B - |V_{r,B}|t}\right) = \frac{\theta_0}{2}.$$

For the second moment of time

$$\arctan\left(\frac{|V_{\tau,A}|t}{r_A - |V_{r,A}|t}\right) + \arctan\left(\frac{|V_{\tau,B}|t}{r_B - |V_{r,B}|t}\right) = \frac{3\theta_0}{2}.$$

Marking Scheme:

1. Proper geometric model, including positions and velocities of the stars — **4 pt.**
2. Determining the tangential velocity of each star — **2 pt.** + **2 pt.**
3. Formulation of the equation(s) for the geometric model — **4 pt.**
   Points may be awarded even if the model is incorrect.
4. Solution of the equation(s) — **6 pt.**
   The numerical method used is not described — -2 pt.
   0 points if the model is incomplete (it is impossible to obtain a meaningful answer).
5. Numerical answers — **1 pt.** + **1 pt.**
   The acceptable range is $\pm 5\,\%$ each.

Note. A participant could also say that radial velocities are too small to make a difference in answers and simply transform the last equation for the first moment of time to $\frac{1}{2}\theta_0 = (|\mu_{\alpha,A}| + |\mu_{\alpha,B}|)\,t$. The answer in this approximation is $t \approx 6.3 \times 10^4$ years. It is out of the 5% boundary specified in the marking scheme. If the participant proves that the radial velocities are small, then points are awarded for the understanding of limitiations of the model. Otherwise, no more than 6 points are awarded for simply using the proper motions.

2 Satellite Tracking

A satellite is moving in a circular orbit around the Earth with an inclination of 20° and an orbital radius of 30 000 km. At noon, the satellite passes through the zenith at a point P_1 with coordinates $(0^\circ\,\mathrm{N},\ 10^\circ\,\mathrm{E})$.

a) At which point P_2 of the Earth will the satellite pass through the zenith after 3 hours?

b) At what astronomical altitude will it be for the observer at P_1 while passing over P_2?

Solution:

a) Since the inclination is positive and does not exceed 90°, the orbital motion of the satellite occurs in the same direction as the rotation of the Earth around its axis. We should consider two cases, since it is not known whether the satellite rises above the plane of the equator or falls below the equator.

Let us consider the case when the satellite rises above the equator. First, we will determine by Kepler's third law how many degrees of its orbit ($r = 3.0 \cdot 10^4$ km) the satellite will fly during the time $t = 3$ h:

$$\frac{T^2}{r^3} = \frac{4\pi^2}{G\mathfrak{M}_\oplus} \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{r^3}{G\mathfrak{M}_\oplus}} = 2\pi\sqrt{\frac{r^3}{g_\oplus R_\oplus^2}} =$$

$$= 2 \times 3.14 \times \sqrt{\frac{(3.0 \cdot 10^7\ \mathrm{m})^3}{9.81\ \mathrm{m/s^2} \times (6.371 \cdot 10^6)^2}} = 5.17 \cdot 10^4\ \mathrm{s} = 14.4^\mathrm{h};$$

$$S_1S_2 = \frac{t}{T} \cdot 360^\circ = \frac{3^\mathrm{h}}{14.4^\mathrm{h}} \times 360^\circ = 75^\circ.$$

The satellite has completed less than a quarter of its orbit.

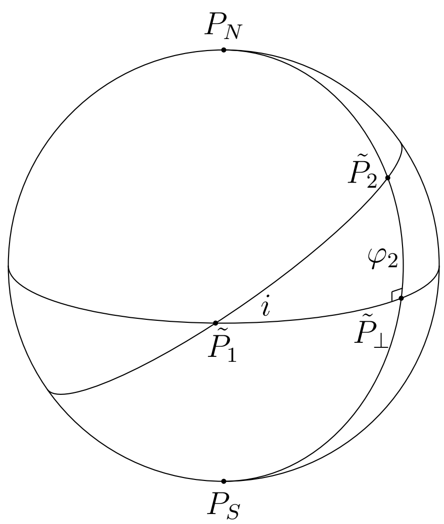


Figure 2: The satellite's motion around the Earth. Ascending node case

Now we assume that the Earth does not rotate — we will take this effect into account when obtaining the final longitude of the second point.

Let $\tilde{P}_1$ and $\tilde{P}_2$ be the projections of positions of the satellites S_1 and S_2 on the non-rotating Earth (see Figure 2). The goal is to find the coordinates $(\varphi_2;\ \tilde{\lambda}_2)$ of the point $\tilde{P}_2$.

We draw a perpendicular from the point $\tilde{P}_2$ to the equator. Using spherical laws for $\triangle\tilde{P}_1\tilde{P}_2\tilde{P}_\perp$ one can write

$$\begin{aligned}&\cos\tilde{P}_1\tilde{P}_2 = \cos\tilde{P}_1\tilde{P}_\perp\cos\tilde{P}_2\tilde{P}_\perp + \sin\tilde{P}_1\tilde{P}_\perp\sin\tilde{P}_2\tilde{P}_\perp\cos\tilde{P}_\perp\\ &\implies \quad \cos\tilde{P}_1\tilde{P}_2 = \cos(\tilde{\lambda}_2-\tilde{\lambda}_1)\cos\varphi_2;\end{aligned}$$

$$\frac{\sin\tilde{P}_1\tilde{P}_2}{\sin\tilde{P}_\perp} = \frac{\sin\tilde{P}_2\tilde{P}_\perp}{\sin\tilde{P}_1} \quad\implies\quad \sin\tilde{P}_1\tilde{P}_2\cdot\sin i = \sin\varphi_2;$$

$$\begin{aligned}&\sin\tilde{P}_2\tilde{P}_\perp\cos\tilde{P}_\perp = \sin\tilde{P}_1\tilde{P}_\perp\cos\tilde{P}_1\tilde{P}_2 - \cos\tilde{P}_1\tilde{P}_\perp\sin\tilde{P}_1\tilde{P}_2\cos\tilde{P}_1\\ &\implies \quad \tan(\tilde{\lambda}_2-\tilde{\lambda}_1) = \tan\tilde{P}_1\tilde{P}_2\cdot\cos i.\end{aligned}$$

The longitude $\tilde{\lambda}_2$ is determined unambiguously

$$\tilde{\lambda}_2 = \tilde{\lambda}_1 + \arctan(\tan\tilde{P}_1\tilde{P}_2\cdot\cos i) = 10^\circ + \arctan(\tan 75^\circ\cdot\cos 20^\circ) = 84^\circ.$$

In 3 hours, the Earth will turn in the same direction as the satellite, at an angle

$$\Delta\lambda = \frac{3^{\mathrm{h}}}{24^{\mathrm{h}}}\times 360^\circ = 45^\circ.$$

Then the actual longitude of the point P_2 is $\lambda_2 = \tilde{\lambda}_2 - \Delta\lambda = 84^\circ - 45^\circ = 39^\circ\,\mathrm{E}$.

The latitude of P_2 is $\sin\varphi_2 = \arcsin(\sin\tilde{P}_1\tilde{P}_2\cdot\sin i) = \arcsin(\sin 75^\circ\sin 20^\circ) = 19.3^\circ$.

Recalling that there are two cases of satellite motion (the starting point could be an ascending or descending node of the orbit), we write down the possible coordinates of point P_2:

$$\begin{aligned}&(19^\circ\,\mathrm{N},\ 39^\circ\,\mathrm{E}),\\ \text{or}\quad &(19^\circ\,\mathrm{S},\ 39^\circ\,\mathrm{E}).\end{aligned}$$

b) First, we determine the angular distance between the observation points:

$$\begin{aligned}\cos P_1P_2 &= \sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2\cos(\lambda_2-\lambda_1) =\\ &= \sin 0^\circ\sin 19^\circ + \cos 0^\circ\cos 19^\circ\cos(39^\circ - 10^\circ) = 0.827\end{aligned}$$

$$P_1P_2 = \arccos 0.827 = 34^\circ.$$

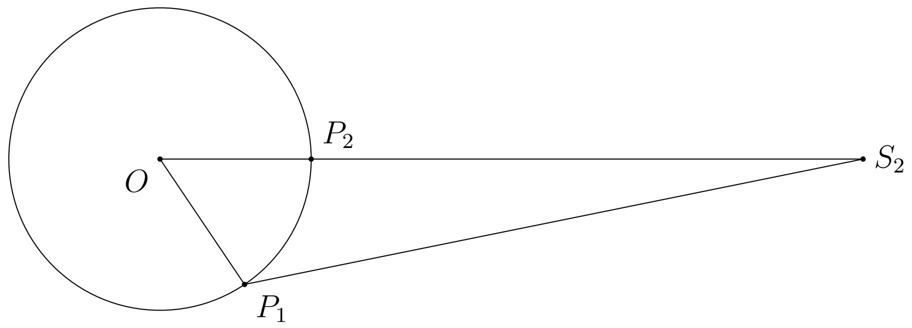


Figure 3: The satellite passes over P_2

Next, we determine the distance from point P_1 to the satellite (point S_2) by the law of cosines (see Figure 3):

$$P_1S = \sqrt{P_1O^2 + SO^2 - 2P_1O \cdot SO \cdot \cos P_1OS} =$$
$$= \sqrt{6\,371^2 + 30\,000^2 - 2 \times 6\,371 \cdot 30\,000 \times \cos 34^\circ}\ [\text{km}] = 2.5 \cdot 10^4\ \text{km}.$$

Finally, we determine the angle OP_1S:

$$OP_1S = \arccos \frac{OP_1^2 + P_1S^2 - OS^2}{2OP_1 \cdot P_1S} =$$
$$= \arccos \frac{6\,371^2 + 2.5 \cdot 10^4 - 30\,000^2}{2 \times 6\,371 \times 2.5 \cdot 10^4} = \arccos(-0.736) \approx 137^\circ.$$

We conclude that from the point P_1, the satellite is observed at the altitude of $137^\circ - 90^\circ = 47^\circ$.

Marking Scheme:

- Question (a)
  1. Arc S_1S_2 that the satellite passes in 3 hours — **2 pt.**
  2. Geometric model, mentioning two cases — **2 pt. + 1 pt.**
  3. Expressions for $(\varphi_2;\ \tilde{\lambda}_2)$ — **4 pt.**
  4. Taking into account the rotation of the Earth — **3 pt.**
  5. Coordinates of P_2 — **2 pt.**
- Question (b)
  6. Geometric model — **2 pt.**
  7. Angular distance between P_1 and P_2 — **1 pt.**
  8. Distance P_1S_2 — **1 pt.**
  9. Altitude of S_2 at P_1 — **2 pt.**

3 Planet on the Spot

There is a large round starspot with a temperature of 2900 K near the center of the disk of a far away star with a radius of $0.7R_\odot$ and a temperature of 3400 K. The decrease in brightness due to the presence of the spot turns out to be the same as during the central transit of a planet with a radius of $6 \cdot 10^4$ km. The axis of rotation of the star is perpendicular to the line of sight, the rotation period is 20 earth days.

a) Estimate the radius of the spot.

b) Estimate the duration of the central transit of the planet through the spot if the radius of the planet's orbit is 0.6 au. The planet's orbit lies in the plane of the star's equator.

Solution:

a) First, we express how the amount of radiation from the star changes in the presence of a spot and at the time of the planet's transit.

If there is a spot of the radius R_s on the disk of the star, instead of some area of the surface of the star with a temperature of Θ, we see a spot with a temperature of Θ_s. Using the Stefan–Boltzmann law we get

$$\Delta E_s = \pi R_s^2 \sigma \Theta^4 - \pi R_s^2 \sigma \Theta_s^4 = \pi R_s^2 \sigma \left(\Theta^4 - \Theta_s^4\right).$$

The planet covers a part of the disk of a star with a radius of R_p, so $\Delta E_p = \pi R_p^2 \sigma \Theta^4$.

The changes in visible brightness are the same, therefore the amounts of energy lost in both events are the same:

$$\pi R_s^2 \sigma \left(\Theta^4 - \Theta_s^4\right) = \pi R_p^2 \sigma \Theta^4,$$

$$\implies \quad R_s = R_p \cdot \frac{\Theta^2}{\sqrt{\Theta^4 - \Theta_s^4}} = 6 \cdot 10^4 \text{ km} \times \frac{3400^2}{\sqrt{3400^4 - 2900^4}} = 8.7 \cdot 10^4 \text{ km}.$$

Such large spots, although irregularly shaped, are formed on BY Draconis type stars.

b) Usually, the direction of the giant planet's orbital motion coincides with the direction of the star's rotation. Let us estimate the relative velocity of the spot and the planet.

The velocity of the spot is determined by the radius and rotation period of the star.

$$V_s = \frac{2\pi R}{T} = \frac{2\pi \times (0.7 \times 7 \cdot 10^8 \text{ m})}{20 \times 86400 \text{ s}} = 1.8 \cdot 10^3 \text{ m/s}.$$

The temperature and the radius determine the M spectral type star. For such stars, we may take the estimation of the star mass at $0.6\,\mathfrak{M}_\odot$. Then the orbital velocity of the planet is

$$V_p = \sqrt{\frac{G\mathfrak{M}}{a}} = \sqrt{\frac{0.6 G\mathfrak{M}_\odot}{0.6 a_\oplus}} = V_\oplus = \frac{2\pi a_\oplus}{T_\oplus} = 3 \cdot 10^1 \text{ km/s}.$$

The mass estimation could also be obtained based on the mass-luminosity relation $L \propto \mathfrak{M}^4$ for the main sequence stars (but this ratio is rather rough for such low-mass stars). Since the temperature and the radius are known, the star luminosity can be estimated from the Stefan–Boltzmann law:

$$L = 4\pi R^2 \sigma \Theta^4 = \\ = 4\pi \times (0.7 \times 7 \cdot 10^8 \text{ m})^2 \cdot 5.67 \cdot 10^{-8} \text{ W/m}^2/\text{K}^4 \times (3400 \text{ K})^4 = 2.3 \cdot 10^{25} \text{ W} = 0.06 L_\odot\,.$$

Taking into account the mass-luminosity relation, we get $\mathfrak{M} = 0.5\mathfrak{M}_\odot$.

The velocities of the planet and the spot differ by more than an order of magnitude, so for an estimate, we can consider the relative velocity equal to the velocity of the planet. From the first to the last contact, the planet has to cover the distance

$$D = 2R_p + 2R_s = 3 \cdot 10^5 \text{ km}.$$

This will take a time interval

$$\Delta t = \frac{D}{V_p} = \frac{3 \cdot 10^5 \text{ km}}{3 \cdot 10^1 \text{ km/s}} = 1 \cdot 10^4 \text{ s} \approx 3^{\text{h}}.$$

Note that the transit time is very short, so we can ignore the change in the apparent shape of the spot when the star's disk rotates.

Marking Scheme:

1. Starspot size estimation — **6 pt.**
2. Planet orbital velocity — **5 pt.**
   Acceptable range for the star's mass is $[0.3; 0.8]\,\mathfrak{M}_\odot$
3. Velocity of the spot — **4 pt.**
4. Transit duration — **5 pt.**

Note. The problem text did not specify which configurations are considered as the beginning and the end of the transit. For example, no penalty is applied if the participant considers the time interval between inner contacts (from the second to the third):

$$\tilde{D} = 2(R_s - R_p) = 5.4 \cdot 10^4 \text{ km},$$

$$\Delta\tilde{t} = \frac{\tilde{D}}{V_p} = 1.8 \cdot 10^3 \text{ s} = 0.5^{\text{h}}.$$

4 Let the Sky Fall

When the sun rises in the west and sets in the east, when the seas go dry and mountains blow in the wind like leaves... (George R. R. Martin)

a) What should be the minimum eccentricity of the Earth's orbit so that the Sun sometimes rises in the west? Consider the semi-major axis of the Earth's orbit and the sidereal day to remain the same as today. Neglect the obliquity of the ecliptic.

b) Estimate the maximum surface temperature of the Earth when the Sun rises in the west. Consider the spherical albedo of the Earth to be equal to 0.3.

c) Can the Earth have an atmosphere under such conditions? Prove your answer with calculations.

Solution:

a) The Sun will move in the opposite direction along the celestial sphere if the angular velocity of the planet's orbital motion around the Sun is greater than the angular velocity of the planet's rotation.

The maximal angular orbital velocity is reached at the point of perihelion:

$$\omega_\mathrm{p} = \frac{v_\mathrm{p}}{r_\mathrm{p}} = \sqrt{\frac{G\mathfrak{M}_\odot}{a_\oplus} \cdot \frac{1+e}{1-e} \cdot \frac{1}{a_\oplus(1-e)}} = \sqrt{\frac{G\mathfrak{M}_\odot}{a_\oplus^3} \cdot \frac{1+e}{(1-e)^3}} = \omega_\oplus^\mathrm{orb}\sqrt{\frac{1+e}{(1-e)^3}}\,.$$

Here $\omega_\oplus^\mathrm{orb} = 360°/T_\oplus = 360°/365.26^\mathrm{d} = 0.986°/\mathrm{d}$.

We equate the obtained value to the angular velocity of the Earth's rotation around its axis $\omega_\oplus^\mathrm{rot} = 360°/\tau_\oplus = 360°/23^\mathrm{h}\,56^\mathrm{m}\,04^\mathrm{s} \approx 361°/\mathrm{d}$:

$$361° = 0.986° \cdot \sqrt{\frac{1+e}{(1-e)^3}} \quad \Longrightarrow \quad e = \left(\frac{361}{0.986}\right)^2 \cdot (1-e)^3 - 1 \qquad (0 \leqslant e < 1).$$

We see an increasing function on the left side of the equation and a decreasing one on the right side. So the equation has only one root. The root can be easily found by brute-force, or one could use the iterative method after rewriting the equation in the form

$$e_{n+1} = 1 - \sqrt[3]{(1+e_n) \cdot \left(\frac{0.986}{361}\right)^2}\,,$$

in which it converges to $e \approx 0.975$ really quickly from any initial seed value.

b) At perihelion, the distance between the Earth and the Sun is equal to

$$r_p = a_\oplus \cdot (1-e) = 0.025\ \mathrm{au} = 3.74 \cdot 10^6\ \mathrm{km} \gg R_\odot.$$

We can write the heat balance equation for the Earth at perihelion:

$$\frac{L_\odot}{4\pi r_p^2} \cdot \pi R_\oplus^2 \cdot (1 - A) = 4\pi R_\oplus^2 \sigma \Theta^4$$

$$\implies \quad \Theta = \sqrt[4]{\frac{L_\odot (1-A)}{16\pi r_p^2 \sigma}} = \sqrt[4]{\frac{3.828 \cdot 10^{26}\ \mathrm{W} \times (1 - 0.3)}{16\pi \times 3.74 \cdot 10^9\ \mathrm{m} \times 5.67 \cdot 10^{-8}\ \mathrm{W/(m^2 \cdot K^4)}}} \approx 1.6 \cdot 10^3\ \mathrm{K}.$$

Here we consider that the Earth radiates as a black body (mainly in the infrared part of the spectrum), while in the optical spectral range it reflects $A = 0.3$ of the incoming energy flux.

c) The root mean square (RMS) velocity of an "average" air molecule (its molar mass is $\mu = 0.029$ kg/mol) at this temperature is

$$v = \sqrt{\frac{3\Re\Theta}{\mu}} = \sqrt{\frac{3 \times 8.314\ \mathrm{J/(mol \cdot K)} \times 1.6 \cdot 10^3\ \mathrm{K}}{0.029\ \mathrm{kg/mol}}} \approx 1.2\ \mathrm{km/s} \approx 0.1\, v_{\mathrm{II}}\,,$$

where $v_{\mathrm{II}} = 11.2$ km/s is the escape velocity for the Earth's surface.

The atmosphere is considered stable if the RMS velocity does not exceed $\sim 20\,\%$ of the escape velocity. Thus, the Earth will retain its atmosphere, but the rocks will begin to melt.

Marking Scheme:

- Question (a)
  1. Equality of orbital and rotational angular velocities — **2 pt.**
  2. Equation for e — **3 pt.**
  3. Finding the root — **3 pt.**
- Question (b)
  4. Distance at perihelion — **1 pt.**
  5. Heat balance equation — **2 pt.**
  6. Θ — **3 pt.**
- Question (c)
  7. Characteristic velocity of an air molecule — **2 pt.**
  8. Comparison with v_{II} — **2 pt.**
  9. Proper conclusion — **2 pt.**

5 Almost Circumpolar

Let's consider stars for which it is true at the same time: (i) in their diurnal motion they cross the horizon, and (ii) their upper culmination occurs to the north of the zenith.

a) Express the fraction of such stars as a function of latitude φ. Assume that stars are distributed uniformly over the celestial sphere.

b) Determine what is the maximum possible fraction of such stars and at what latitude this maximum is achieved.

Solution:

a) Let us determine the location of suitable stars on the diagram $(\varphi; \delta)$. The upper culmination of the star occurs to the north of the zenith if $\delta > \varphi$.

A star does not cross the horizon in two cases: either it is circumpolar or it is non-rising. A star is circumpolar if its lower culmination is above the horizon: $h_{\min} = |\varphi + \delta| - 90^\circ \geqslant 0^\circ \implies |\varphi + \delta| \geqslant 90^\circ$. A star is non-rising if its upper culmination is below the horizon: $h_{\max} = 90^\circ - |\varphi - \delta| \leqslant 0^\circ \implies |\varphi - \delta| \geqslant 90^\circ$. Considering $\delta > \varphi$, we find 4 different cases:

Case 1:	$\varphi \leqslant -45^\circ$	$\implies$	$-(90^\circ + \varphi) \leqslant \delta \leqslant 90^\circ + \varphi$;
Case 2:	$-45^\circ \leqslant \varphi \leqslant 0^\circ$	$\implies$	$\varphi \leqslant \delta \leqslant 90^\circ + \varphi$;
Case 3:	$0^\circ \leqslant \varphi \leqslant 45^\circ$	$\implies$	$\varphi \leqslant \delta \leqslant 90^\circ - \varphi$;
Case 4:	$\varphi \geqslant 45^\circ$	$\implies$	$\delta \in \varnothing$.

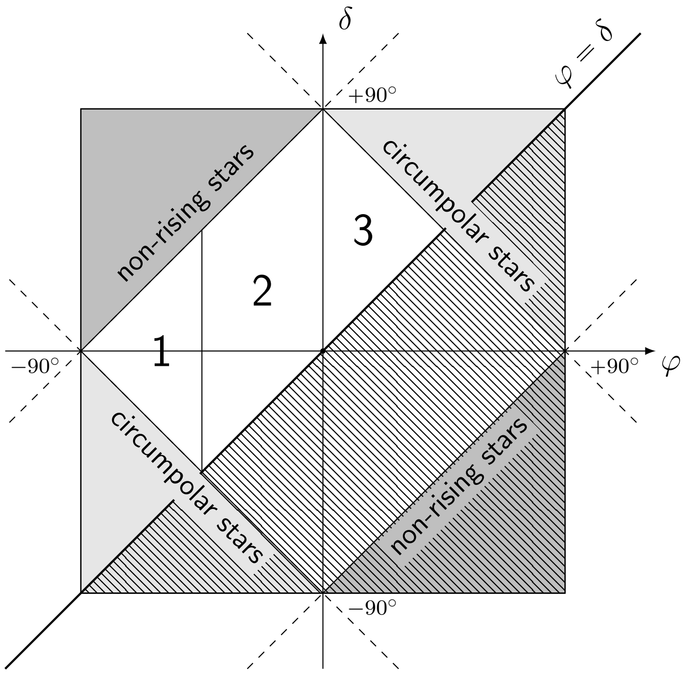


Figure 4: Selecting areas that meet the conditions of the problem

The fraction of the area of a sphere enclosed between small circles with declinations from $\delta_{\min}$ to $\delta_{\max}$ can be calculated using the spherical segment area formula $S_{\delta \geqslant \delta_0} = 2\pi R^2(1 - \sin\delta_0)$:

$$\eta = \frac{2\pi R^2(1-\sin\delta_{\min}) - 2\pi R^2(1-\sin\delta_{\max})}{4\pi R^2} = \frac{\sin\delta_{\max} - \sin\delta_{\min}}{2}.$$

Next, we obtain the fractions for the each case stated above:

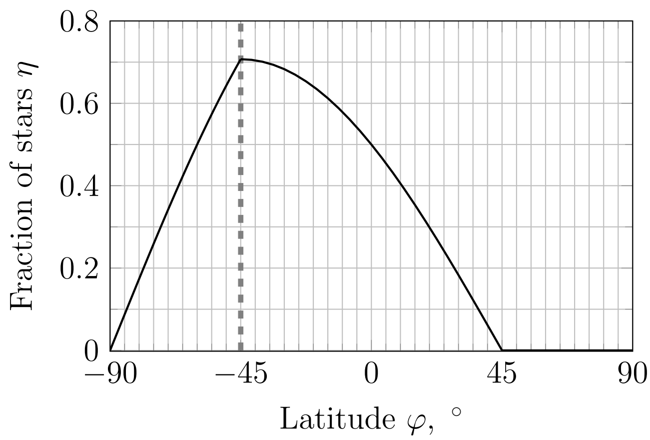
Case 1: $$\eta_1 = \frac{\sin(90^\circ + \varphi) - \sin(-90^\circ - \varphi)}{2} = \cos\varphi;$$

Case 2: $$\eta_2 = \frac{\sin(90^\circ + \varphi) - \sin\varphi}{2} = \frac{\cos\varphi - \sin\varphi}{2} = \frac{\sin(45^\circ - \varphi)}{\sqrt{2}};$$

Case 3: $$\eta_3 = \frac{\sin(90^\circ - \varphi) - \sin\varphi}{2} = \frac{\cos\varphi - \sin\varphi}{2} = \frac{\sin(45^\circ - \varphi)}{\sqrt{2}}.$$

Finally,

$$\eta(\varphi) = \begin{cases} \cos\varphi, & \varphi \leqslant -45^\circ; \\ \frac{1}{\sqrt{2}}\sin(45^\circ - \varphi), & -45^\circ \leqslant \varphi \leqslant 45^\circ; \\ 0, & \varphi \geqslant 45^\circ. \end{cases}$$



b) Obviously, the maximum of η is achieved at $\varphi = -45^\circ$ and equals

$$\cos(-45^\circ) = \frac{\sin(45^\circ - (-45^\circ))}{\sqrt{2}} = \frac{1}{\sqrt{2}} \approx 0.71.$$

Marking Scheme:

- Question (a)
  1. The presence of a graph or formulas for culminations — **3 pt.**
  2. Indication of 4 possible cases — **4** × **1 pt.** = **4 pt.**
  3. Relationship of latitude and declination for each case — **4** × **1 pt.** = **4 pt.**
  4. Calculation of the spherical area — **2 pt.**
  5. Providing answers for three possible cases — **3** × **1 pt.** = **3 pt.**
- Question (b)
  6. Discussion of the maximum value — **2 pt.** ([−45°; 45°]) + **1 pt.** ([−90°; −45°])
  7. Correct final answer — **1 pt.**

6 Jupiter in Sirius

Let's observe Jupiter from a point near the Omega Sirius Hotel tonight!

a) In which constellation is Jupiter located?
b) At what civil time and at what altitude the upper culmination of Jupiter does occur? Neglect the equation of time.
c) Estimate the period of visibility of Jupiter during the current day.
d) What is the elongation of Jupiter?
e) What is the angular size of Jupiter?
f) Estimate the date of the nearest opposition of Jupiter in the future.

Date		16 September 2024
Jupiter	Right ascension α	$5^h\,17^m\,28^s$
	Declination δ	$+22^\circ\,22.0'$
Omega Sirius	Latitude φ	$43^\circ\,24.7'$ N
	Longitude λ	$39^\circ\,57.0'$ E
	Time zone	Moscow (UT+3)

Solution:

a) The analysis of the equatorial coordinates makes it clear that Jupiter is located on the ecliptic approximately at $43^m \approx 11^\circ$ west of the summer solstice point ($\alpha = 6^h$, $\delta = +23^\circ\,26'$). Since the summer solstice point is located in the Taurus constellation, almost on the border with the Gemini constellation, Jupiter is also located in Taurus.

b) The altitude of Jupiter's upper culmination is

$$h = 90^\circ - \varphi + \delta = +69^\circ.$$

On the day of the Northern hemisphere autumnal equinox, September 22, the solar and sidereal times are equal. On September 16, 6 days before the equinox, the sidereal time is $\Delta t = 3^m\,56^s \times 6 \approx 0.4^h$ behind solar time. Thus, the upper culmination of Jupiter occurs at $\alpha + \Delta t \approx 5.7^h$ local solar time. That is

$$5.7^h - \lambda = 5.7^h - \frac{39^\circ\,57.0'}{15^\circ/\mathrm{h}} = 5.7^h - 2.7^h = 3.0^h \text{ UT},$$

or $3.0^h + 3^h = 6.0^h$ civil time.

c) Next, we determine the hour angle at the moment of Jupiter's set:

$$\cos t = -\tan\varphi \cdot \tan\delta$$
$$\implies \quad t = \arccos\left(-\tan 43.41^\circ \cdot \tan 22.37^\circ\right) = \arccos(-0.389) \approx 113^\circ \approx 7.5^h.$$

Thus, Jupiter crosses the horizon at $6.0^\mathrm{h} \pm 7.5^\mathrm{h}$ civil time. It rises at 22.5^h and sets at 13.5^h. Here the difference in the duration of the solar and sidereal days can be neglected due to the reasonable accuracy of calculations.

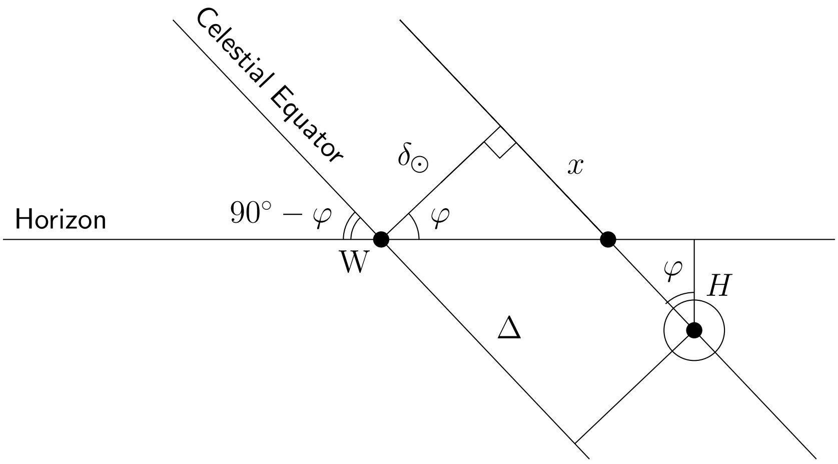


Figure 5: Accounting for small corrections when determining the moment of sunset

The Sun interferes with observations. Near the equinox, the declination of the Sun changes at a rate of about $1^\circ \cdot \sin\varepsilon \approx 0.4^\circ$ per day. On September 16, $\delta_\odot = 0.4^\circ \times 6 \approx +2.4^\circ$.

On this day, the center of the Sun crosses the horizon at $12^\mathrm{h} + x$ after the local noon (see Figure 5), where

$$x = \delta_\odot \cdot \tan\varphi \approx 2.3^\circ = 9.2^\mathrm{m}.$$

In addition, the atmospheric refraction ($r = 35'$) and the angular radius of the Sun ($\rho = 16'$) extend the daylength by

$$\frac{H}{\cos\varphi} = \frac{r+\rho}{\cos\varphi} \approx 1.2^\circ = 4.8^\mathrm{m}$$

both at sunrise and sunset. In total, we have a correction $\Delta = 9.2^\mathrm{m} + 4.8^\mathrm{m} = 14^\mathrm{m}$.

The Sun rises at

$$6.0^\mathrm{h} - \lambda + 3^\mathrm{h} + \Delta = 6.0^\mathrm{h} + 20^\mathrm{m} - 14^\mathrm{m} = 6^\mathrm{h}\,06^\mathrm{m} = 6.1^\mathrm{h},$$

and sets at

$$18.0^\mathrm{h} + 20^\mathrm{m} + 14^\mathrm{m} = 18^\mathrm{h}\,34^\mathrm{m} \approx 18.5^\mathrm{h}.$$

Thus, Jupiter can be observed approximately since its rise at 22.5^h until its upper culmination and the sunrise at approximately 6^h in the morning, 7.5^h in total.

d) The elongation of Jupiter is the angular distance between the Sun and Jupiter as observed from the Earth. It can be estimated as the difference of the ecliptic longitudes of the Sun (near the autumnal equinox) and Jupiter (near the summer solstice):

$$\angle SEJ = \lambda_\odot - \lambda_J \approx \left(180^\circ - 6^\mathrm{d} \times 1^\circ/\mathrm{d} \times \cos\varepsilon\right) - \left(90^\circ - \frac{6^\mathrm{h} - \alpha_J}{\cos\varepsilon}\right) \approx 96^\circ \text{ west of the Sun.}$$

It is also possible to estimate the elongation of Jupiter as the angular distance between the Sun and Jupiter for an observer on the Earth. The equatorial coordinates of Jupiter are given in the table. The equatorial coordinates of the Sun we can estimate based on the date of observation. The declination of the Sun was calculated earlier ($\delta_\odot = 2.4°$). The right ascension near the autumnal equinox can be estimated approximately using the formula

$$\alpha_\odot = 180° - \frac{\delta_\odot}{\tan\varepsilon} = 180° - \frac{2.4°}{\tan 23.4°} = 174.5° = 7.3^{\mathrm{h}}.$$

The elongation of Jupiter can be estimated using the sprerical law of cosines

$$\cos(\lambda_\odot - \lambda_J) = \sin\delta_J \sin\delta_\odot + \cos\delta_J \cos\delta_\odot \cos(\alpha_J - \alpha_\odot) =$$
$$= \sin 22.4° \sin 2.4° + \cos 22.4° \cos 2.4° \cos(79.4° - 174.5°) = -0.066;$$

$$\lambda_\odot - \lambda_J \approx 94°.$$

e) Jupiter is near the quadrature, so the distance to the planet can be estimated by the Pythagorean theorem: $d = \sqrt{5.2^2 - 1.0^2} \approx 5.1$ au. The angular size of Jupiter is

$$\frac{2R_J}{d} = \frac{2 \times 69.9 \cdot 10^3 \text{ km}}{5.1 \times 149.6 \cdot 10^6 \text{ km}} = 1.83 \cdot 10^{-4} \text{ rad} \approx 38''.$$

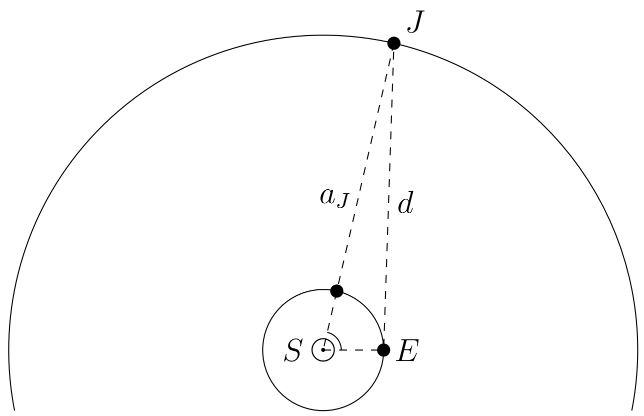


Figure 6: Jupiter, the Earth and the Sun at the moment of observation

f) Since Jupiter is located to the west of the Sun, the Earth "catches up" with it in its orbital motion. For the opposition to occur, the Earth has to shift relatively to the Sun–Jupiter direction by

$$\angle JSE = \arcsin\left(\sin\angle SEJ \cdot \frac{d}{a_J}\right) \approx 77°.$$

This shift is associated with the relative motion of the planets and with the synodic period of Jupiter relatively to the Earth.

The synodic period of Jupiter

$$S = \frac{T_J \cdot T_\oplus}{T_J - T_\oplus} = \frac{11.86 \times 1.00}{11.86 - 1.00} \,[\text{yr}] = 1.09 \text{ yr},$$

so the opposition will come in

$$\frac{77^\circ}{360^\circ} \times 1.09 \text{ yr} = 0.233 \text{ yr} \approx 85 \text{ days}.$$

That is about December 10, 2024. The actual opposition of Jupiter will occur on December 7.

Marking Scheme:

1. Question (a) — **2 pt.**

- Question (b)
  2. Altitude — **2 pt.**
  3. Civil time — **3 pt.**

4. Question (c) — **4 pt.**
5. Question (d) — **3 pt.**
6. Question (e) — **3 pt.**
7. Question (f) — **3 pt.**

7 Yet Another Pinwheel

The radial surface-brightness profile of disc galaxies is overall well described by an exponential function:

$$I(R) = I_0 \exp\left(-\frac{R}{h}\right),$$

where R is the distance from the center of the galaxy, h is the radial scale length and I_0 is the central surface brightness measured in $L_\odot/\text{pc}^2$.

In the spectrum of the observed face-on disc galaxy (the axis of the galaxy coincides with the line of sight), the center of the H_α line is observed at a wavelength of 6670 Å. The apparent magnitude of the galaxy is 14^m and h is assumed to be 3 kpc.

a) Estimate the effective radius R_e of the galaxy. R_e is the radius within which 50 % of the total light of a galaxy is emitted.

b) Estimate the central surface brightness I_0.

c) Estimate the baryonic mass of the galaxy.

Solution:

a) First, we derive the formula for the total luminosity of the galaxy. It is necessary to integrate the surface brightness on the radius throughout the visible disk of the galaxy. When transforming the area integration to radius integration, a multiplier of $2\pi R$ appears: due to circular symmetry, the apparent disk of the galaxy can be divided into rings with a width of dR and an area of $2\pi R\,dR$ with constant surface brightness, and the luminosities from such rings can be summed:

$$L = \int\limits_S I(R)\,dR = \int\limits_0^{+\infty} I(R)\cdot 2\pi R\,dR = 2\pi I_0 \int\limits_0^{+\infty} Re^{-\frac{R}{h}}\,dR\,;$$

$$\int Re^{-\frac{R}{h}}\,dR \xlongequal{\tau\equiv\frac{R}{h}} h^2\int \tau e^{-\tau}\,d\tau = h^2\left(-\tau e^{-\tau} + \int e^{-\tau}\,d\tau\right) = -h^2(\tau+1)e^{-\tau} + \text{const}$$

$$\Longrightarrow \quad L = -2\pi I_0\cdot h^2\left(\frac{R}{h}+1\right)e^{-\frac{R}{h}}\Bigg|_{h=0}^{+\infty} = 2\pi I_0 h^2.$$

Next, we should find the value of the radius R_e within which the half of the total luminosity, or $\pi I_0 h^2$, is emitted:

$$\pi I_0 h^2 = \int\limits_0^{R_e} I(R)\cdot 2\pi R\,dR = -2\pi I_0 h^2\left(\frac{R}{h}+1\right)e^{-\frac{R}{h}}\Bigg|_{h=0}^{R_e}$$

$$\Longrightarrow \quad \left(\frac{R_e}{h}+1\right)e^{-\frac{R_e}{h}} \equiv (\tau_e+1)\,e^{-\tau_e} = \frac{1}{2}.$$

For physical reasons (due to non-decreasing luminosity with increasing radius), the equation

$$f(\tau) = \frac{1}{2} - (\tau + 1)\, e^{-\tau} = 0$$

should have only one root $\tau = \tau_e$. Beyond a few radial scale lengths, the surface brightness becomes extremely low, so we can choose a smaller interval for searching the root.

We look for the root on the interval [0 : 10] using the bisection method:

τ	0	10	5	2	1	1.5	1.7
$f(\tau)$	−0.50	0.50	0.46	0.09	−0.23	−0.06	0.007

Therefore, $R_e \approx 1.7h \approx 5$ kpc.

b) Let us determine the luminosity of the galaxy. To do this, we should estimate the distance D to the galaxy. Knowing the value of the displacement of the center of the spectral line, we can estimate the galaxy redshift and apply Hubble's law:

$$z = \frac{v}{c} = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0} = \frac{6670\ \text{Å} - 6563\ \text{Å}}{6563\ \text{Å}} = 0.016;$$

$$V = cz = 3.0 \cdot 10^5\ \text{km/s} \times 0.016 = 4.9 \cdot 10^3\ \text{km/s};$$

$$D = \frac{V}{H_0} = \frac{4.9 \cdot 10^3\ \text{km/s}}{70\ (\text{km/s})/\text{Mpc}} = 70\ \text{Mpc}.$$

The redshift is quite small, and its effect on luminosity is negligible. So we can calculate the absolute magnitude of the galaxy to retrieve its luminosity and central surface brightness:

$$M = m + 5 - 5 \lg r = 14 + 5 - 5 \lg(7 \cdot 10^7) = -20.2^{\text{m}};$$

$$L = 10^{-0.4(M - M_\odot)} L_\odot = 9.6 \cdot 10^9\, L_\odot$$

$$\Longrightarrow \quad I_0 = \frac{L}{2\pi h^2} = \frac{9.6 \cdot 10^9 L_\odot}{2\pi \times (3 \cdot 10^3\ \text{pc})^2} = 1.7 \cdot 10^2\, L_\odot/\text{pc}^2.$$

c) It is necessary to estimate the baryonic mass from the known galaxy luminosity. The galaxy is spiral and fainter than the Milky Way ($M_{\text{MW}} \approx -21.3^{\text{m}}$). We don't know much about this galaxy, but we can assume that the parameters of the galaxy are proportional to the parameters of the Milky Way.

The luminosity of the galaxy is $L = 10^{0.4(-21.3+20.2)} L_{\text{MW}} \approx 0.4 L_{\text{MW}}$.

Assuming the mass-to-light ratio of the galaxy and the Milky Way are the same, we obtain the total mass of the galaxy $\sim 0.4\mathfrak{M}_{\text{MW}} \sim 10^{12}\mathfrak{M}_\odot \times 0.4 = 4 \cdot 10^{11}\mathfrak{M}_\odot$. Also we recall that the baryonic mass in the Milky Way is about 10 % of the total mass. Then the baryonic mass of the galaxy is $4 \cdot 10^{10} M_\odot$.

Another way is to use $L/\mathfrak{M} \sim L_\odot/\mathfrak{M}_\odot$. Thus, $\mathfrak{M} \sim 10^{10}\, \mathfrak{M}_\odot$ (rough but effective).

Marking Scheme:

- Question (a)
  1. Expression of the total luminosity using an integral; $L = 2\pi I_0 h^2$ — **2 pt.** + **2 pt.**
  2. Expression for R_e; simplified equation for R_e or τ_e — **1 pt.** + **2 pt.**
  3. Estimation of the root of $f(\tau) = 0$ — **3 pt.**
     If the root of $I(R) = I_0/2$ is estimated instead of R_e, the total score for Question (a) cannot exceed 1 pt.
- Question (b)
  4. Estimation of the distance to the galaxy — **2 pt.**
  5. Estimation of the luminosity — **2 pt.**
  6. Estimation of the central surface brightness — **3 pt.**

6. Question (c) — **3 pt.**
   If the presence of dark matter is not taken into account or an incorrect mass-to-light ratio is used, no more than 2 pt. can be awarded for Question (c).

8 Thirty Degrees of Freedom

A spherical molecular cloud has a radius of 5 pc and a mass of $2 \cdot 10^3\,\mathfrak{M}_\odot$. The temperature of the gas is 30 K, the cloud is homogeneous.

a) Estimate the mean free path and the mean free time (between successive collisions) of the hydrogen molecules in the cloud. The cross section radius of H_2 is 2.7 Å.

b) Let a star pass behind the cloud along its apparent diameter with a tangential velocity of 70 km/s. For how long will the absorption be greater than 0.5^m? Absorption in the V band is related to the atomic column density by formula $A_V = 5.2 \cdot 10^{-22}\ N_H$ in magnitudes, where N_H is measured in cm^{-2}.

Solution:

a) The mean free path of the molecules is $l_{\text{H}_2} = \dfrac{1}{\sqrt{2}\pi d^2 n}$, where πd^2 is the collisional cross section of a molecule, $d = 2.7$ Å is the cross section radius. The $\sqrt{2}$ coefficient arises from taking into account the motion of molecules relative to each other. Note, however, that taking the latter into account does not critically affect the estimations of the values needed for the solution.

The concentration of the molecules in the cloud

$$n = \frac{N}{V} = \frac{M/m_{\text{H}_2}}{\frac{4}{3}\pi R^3} = \frac{2 \cdot 10^3 \times 2 \cdot 10^{30}\ \text{kg}/(2 \times 1.67 \cdot 10^{-27}\ \text{kg})}{\frac{4}{3}\pi \times (5 \times 206\,265 \times 1.49 \cdot 10^{11}\ \text{m})^3} \approx 8 \cdot 10^7\ \text{m}^{-3} = 8 \cdot 10^1\ \text{cm}^{-3}.$$

Here the mass of H_2 is estimated as $2m_p$.

$$l_{\text{H}_2} = \frac{1}{\sqrt{2}\pi \times (2.7 \cdot 10^{-10}\ \text{m})^2 \times 8 \cdot 10^7\ \text{m}^{-3}} \approx 4 \cdot 10^{10}\ \text{m} \approx 0.3\ \text{au}.$$

The characteristic thermal velocity of the molecules is

$$v_t = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3\mathfrak{R}T}{\mu}} = \sqrt{\frac{3 \times 8.31\ \text{J}/(\text{mol} \cdot \text{K}) \times 30\ \text{K}}{0.002\ \text{kg/mol}}} \approx 6 \cdot 10^2\ \text{m/s}.$$

We can also estimate the characteristic velocities associated with the gravitational field of the cloud. We may use the virial theorem assuming that the cloud is not collapsing nowadays ($2\langle E_k \rangle + \langle U \rangle = 0$), or calculate the circular velocity at the cloud boundary

$$v_g \sim \sqrt{\frac{G\mathfrak{M}}{R}} = \sqrt{\frac{G\mathfrak{M}_\odot}{a_\oplus} \cdot \frac{2 \cdot 10^3}{206\,265 \times 5}} = v_\oplus \times \sqrt{\frac{2 \cdot 10^3}{206\,265 \times 5}} = 30\ \text{km/s} \times 0.044 \sim 1.3\ \text{km/s}.$$

As a result, to estimate the mean free time we may take the value of velocity of $1.0 \div 1.5$ km/s. The mean free time is

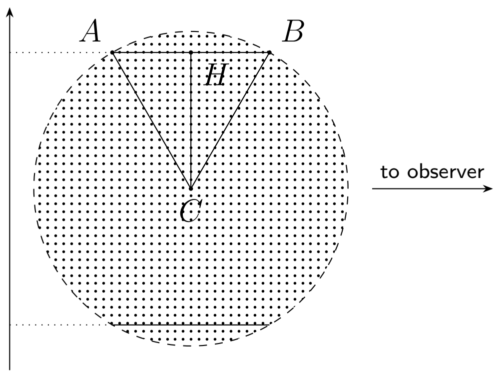
$$t_{\text{H}_2} = \frac{l_{\text{H}_2}}{v_g} \sim \frac{0.3\ \text{au}}{1\ \text{km/s}} \sim 5 \cdot 10^7\ \text{s} \sim 1 \div 2\ \text{yr}.$$

b) We estimate the absorption in the direction of the cloud center, that is, the maximum possible absorption in the cloud:

$$\max A_V = 5.2 \cdot 10^{-22}\ \text{cm}^2 \cdot N_H = 5.2 \cdot 10^{-22}\ \text{cm}^2 \cdot 2n \cdot 2R =$$
$$= 5.2 \cdot 10^{-22}\ \text{cm}^2 \times (2 \times 8 \cdot 10^1\ \text{cm}^{-2}) \cdot (2 \times 5 \times 206\,265 \times 1.49 \cdot 10^{13}\ \text{cm}) \approx 2.6^\text{m}.$$

The minimal path of light through the cloud for the absorption to be at least 0.5^m is

$$AB = 2R \cdot \frac{0.5}{2.6} \approx 2\ \text{pc}.$$



The star has to travel the distance

$$2CH = 2 \cdot \sqrt{5^2 - 0.95^2} \approx 10\ \text{pc},$$

which is approximately the cloud's diameter.

The time needed for that travel

$$\tau \approx \frac{2R}{v_s} = \frac{10 \times 206\,265 \times 1.49 \cdot 10^{11}\ \text{km}}{70\ \text{km/s}} = 4.4 \cdot 10^{12}\ \text{s} \approx 1.4 \cdot 10^5\ \text{yr}.$$

Marking Scheme:

- Question (a)
  1. Concentration of the molecules — **2 pt.** + **1 pt.** (formula + estimation)
  2. Mean free path — **2 pt.** + **1 pt.**
  3. Characteristic velocity of molecules — **2 pt.** + **1 pt.**
  4. Two estimations of the velocity (thermal and gravitational), conclusion — **2 pt.**
  5. Mean free time — **1 pt.**
- Question (b)
  6. Absorption in the direction of the cloud center — **2 pt.** + **1 pt.**
  7. Light travel distance corresponding to an absorption of 0.5^m — **2 pt.**
  8. Length of the star's path — **2 pt.**
  9. Star's travel time — **1 pt.**

9 High Precision Guidance

A radiotelescope RT-22 (CrAO) has a mirror dish with a diameter of 22 m and focal length $F = 9.5$ m. They are going to observe the Sun in the first sidelobe of the radiation pattern of the telescope (see Figure 7) at a frequency of 420 MHz. A receiver is located at the primary (main) focus of the mirror. Consider the mirror to be spherical. Determine the distance between the center of the mirror and the center of the receiver's shadow on the dish.

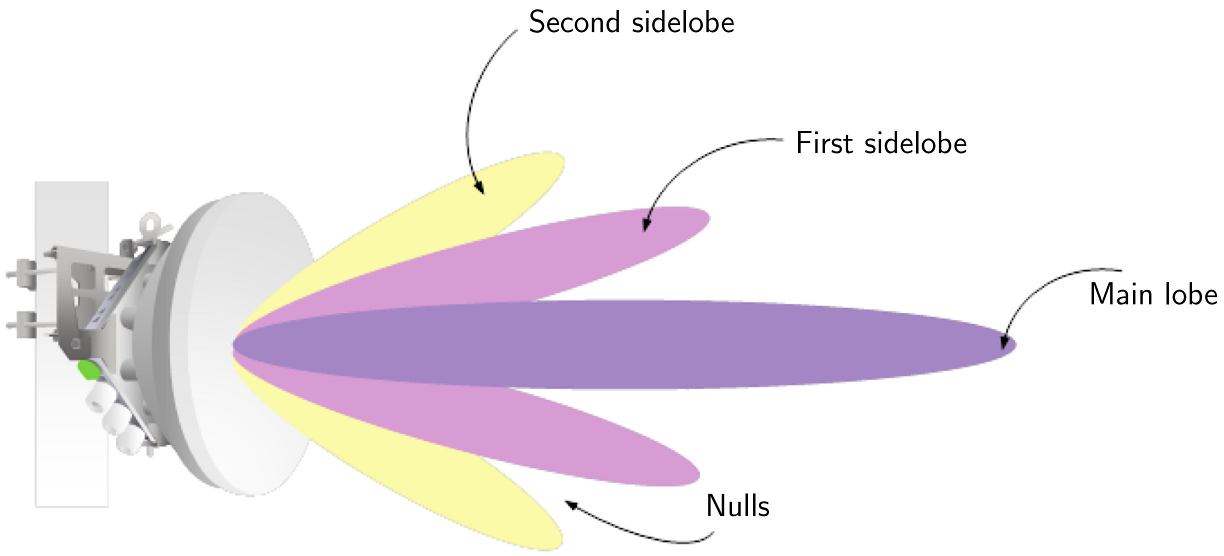


Figure 7: Radiation pattern lobes (beams) of an antenna

Solution:

An image in the focal plane of the dish is determined by the diffraction pattern on a circular aperture (Figure 8). First, we estimate the angular distance of the peak of the first sidelobe relative to the axis of symmetry of the system. Note that to do this, we need to determine the position of the first maximum of the diffraction pattern outside the Airy disc. Recall that the angular radius of the Airy disc is $1.22\lambda/D$, where D is a diameter of the dish and λ is the wavelength of light. Further, the distances between the minima are about λ/D, so the angular distance between the peak of the main lobe (the center of the image) and the peak of the first sidelobe is about $1.22\lambda/D + 0.5\lambda/D \approx 1.7\lambda/D$.

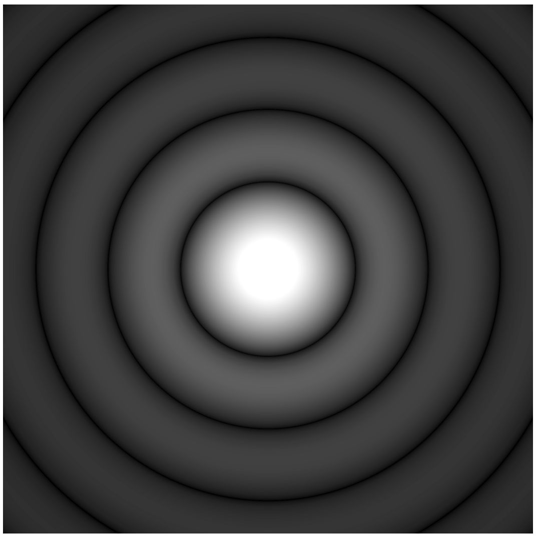


Figure 8: Airy pattern

To estimate the angular distance, we calculate the wavelength at which the observations are carried out:

$$\lambda = \frac{c}{\nu} = \frac{3.00 \cdot 10^8 \text{ m/s}}{420 \cdot 10^6 \text{ Hz}} = 0.714 \text{ m}.$$

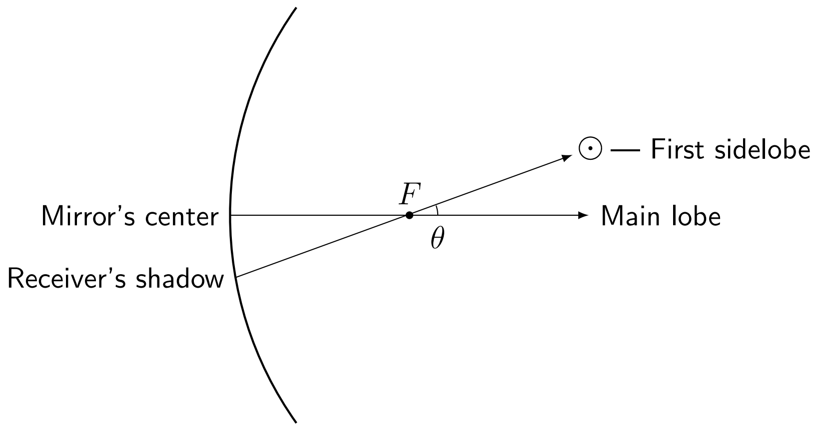
Thus,

$$\theta \approx 1.7 \cdot \frac{\lambda}{D} = 1.7 \times \frac{0.714 \text{ m}}{22 \text{ m}} \approx 0.055 \text{ rad} \approx 3.2^\circ.$$

Note that the angular diameter of the Sun ($\approx 0.5^\circ$) is much smaller than the resolution of the telescope ($\lambda/D \approx 1.9^\circ$), so we may consider the Sun to be a point source.

Such an angular distance corresponds to a linear distance of

$$l = F \tan\theta \approx 9.5 \text{ m} \times 0.055 \text{ rad} \approx 0.5 \text{ m}.$$



Note that the sphericity of the mirror can be ignored. Possible mirror aberrations are proportional to the third power of the angle between the beam and the main optical axis. Taking into account such effects will affect only the fourth digit in the answer. The accuracy of the data given in the problem does not allow one to discuss such effects. The difference in the distance along the surface of the mirror dish and along the chord can also be neglected due to the smallness of the angle obtained.

Marking Scheme:

1. Wavelength λ — **2 pt.**
2. Resolution estimation using the formula $\propto \lambda/D$ — **3 pt.**
3. Coefficient in the formula for the angular distance with justification — **6 pt.**
4. Linear distance based on angular distance — **4 pt.**
5. Rationale for not taking into account the sphericity of the mirror — **5 pt.**
6. *Bonus criterion.* The final answer MUST NOT contain more than two significant digits. For each extra digit the penalty is −2 points.

   1 pt. is awarded in total for the problem in case of some sane attempt to solve it.

Constants

Universal

Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R} = 8.314$ J/(mol · K)
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au = $149.6 \cdot 10^{6}$ km
Parsec	1 pc = 206 265 au
Hubble constant	$H_0 = 70$ (km/s)/Mpc

Earth

Radius	$R_\oplus = 6371$ km
Mass	$\mathfrak{M}_\oplus = 5.97 \cdot 10^{24}$ kg
Obliquity of ecliptic	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26^{\text{d}}$
Orbital eccentricity	$e_\oplus = 0.0167$

Hydrogen spectrum

Lyman Lα	$\lambda_{\text{L}\alpha} = 1215.7$ Å
Balmer Hα	$\lambda_{\text{H}\alpha} = 6562.8$ Å

Jupiter

Radius	$R_J = 6.99 \cdot 10^{4}$ km
Mass	$\mathfrak{M}_J = 1.90 \cdot 10^{27}$ kg
Orbital radius	$a_J = 5.20$ au
Orbital period	$T_J = 11.86$ yr

Sun

Radius	$R_\odot = 6.96 \cdot 10^{5}$ km
Mass	$\mathfrak{M}_\odot = 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot = 4.74^{\text{m}}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^{3}$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Emission constants

Stefan–Boltzmann	$\sigma = 5.67 \cdot 10^{-8}$ (W/m^2)/K^4
Wien's displacement	$b = 2898$ µm · K

UBV system	Mean wavelengths
U band	$\lambda_U = 364$ nm
B band	$\lambda_B = 442$ nm
V band	$\lambda_V = 540$ nm

Number of Siriuses in the Solar system $\geqslant 1$