1 Celestial Pause

From opposition to the nearest stationary point, the apparent angular diameter of an asteroid decreases by 10 %. A stationary point is the point at which the asteroid's apparent motion reverses. The observer is located on the surface of the Earth. Assume the asteroid's orbit is circular and lies in the ecliptic plane.

a) Determine the radius of the asteroid's orbit.

b) How much time passes between these two moments?

Solution:

a) A stationary point is a configuration in which the relative velocity of the observer and the asteroid is directed along the line of sight, meaning there is no relative tangential component. The stationary points are symmetric with respect to opposition.

Let λ denote the difference in the heliocentric longitudes of the Earth and the asteroid. The distance between them is

$$AE = \sqrt{ES^2 + AS^2 - 2ES \cdot AS \cos\lambda} = \sqrt{a_E^2 + a^2 - 2a_E a \cos\lambda}.$$

Express the tangential components of velocity:

$$v_{\perp E} = v_E \sin(\angle SEA - 90^\circ) = -v_E \cos\angle SEA;$$

$$v_{\perp A} = v_A \sin(90^\circ - \angle SAE) = v_A \cos\angle SAE.$$

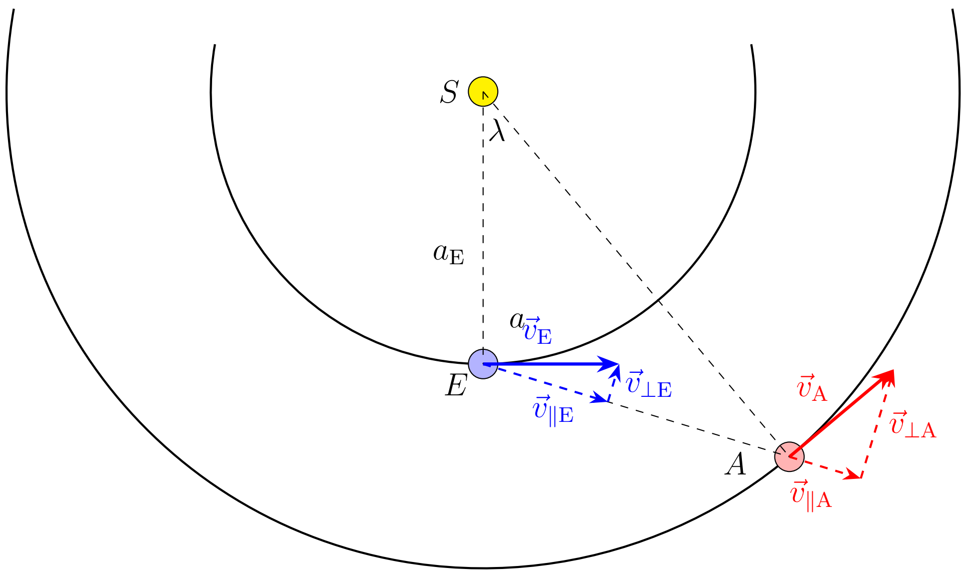


Figure 1: On determining the position of the stationary point

Apply the law of cosines to $\triangle SEA$:

$$SA^2 = SE^2 + AE^2 - 2SE \cdot AE \cos\angle SEA \quad \Longrightarrow \quad \cos\angle SEA = \frac{SE^2 + AE^2 - SA^2}{2SE \cdot AE},$$

$$SE^2 = SA^2 + AE^2 - 2SA \cdot AE \cos\angle SAE \quad \Longrightarrow \quad \cos\angle SAE = \frac{SA^2 + AE^2 - SE^2}{2SA \cdot AE};$$

$$\cos\angle SEA = \frac{a_E - a\cos\lambda}{\sqrt{a_E^2 + a^2 - 2a_E a\cos\lambda}}, \qquad \cos\angle SAE = \frac{a - a_E\cos\lambda}{\sqrt{a_E^2 + a^2 - 2a_E a\cos\lambda}}.$$

Assuming circular orbits, we equate the tangential components of the orbital velocities:

$$-\sqrt{\frac{G\mathfrak{M}_\odot}{a_E}} \cdot \frac{a_E - a\cos\lambda}{\sqrt{a_E^2 + a^2 - 2a_E a\cos\lambda}} = \sqrt{\frac{G\mathfrak{M}_\odot}{a}} \cdot \frac{a - a_E\cos\lambda}{\sqrt{a_E^2 + a^2 - 2a_E a\cos\lambda}},$$

$$\frac{a\cos\lambda - a_E}{\sqrt{a_E}} = \frac{a - a_E\cos\lambda}{\sqrt{a}} \quad \Longrightarrow \quad \cos\lambda = \frac{a\sqrt{a_E} + a_E\sqrt{a}}{a\sqrt{a} + a_E\sqrt{a_E}}.$$

The distance to the asteroid at opposition is $r_o = a - a_E$. The distance at the stationary point is derived using the geometrical relation involving $\cos\lambda$. The ratio of the asteroid's angular sizes at these two points leads to the following equation:

$$(a - a_E)^2 = 0.9^2 \left(a_E^2 + a^2 - 2a_E a \cdot \frac{a\sqrt{a_E} + a_E\sqrt{a}}{a\sqrt{a} + a_E\sqrt{a_E}}\right).$$

Let us measure distances in au, so $a_E = 1$. The equation simplifies to:

$$a^2 - 2a + 1 = 0.81\left(1 + a^2 - 2a \cdot \frac{a + \sqrt{a}}{a\sqrt{a} + 1}\right) \qquad \Longrightarrow \qquad a = 5.6 \text{ au}.$$

b) Next, determine the time interval between opposition and the stationary point. The difference in heliocentric longitudes at the stationary point is

$$\lambda = \arccos\frac{a + \sqrt{a}}{a\sqrt{a} + 1} = 56^\circ.$$

The change in the difference of heliocentric longitudes is related to the synodic period S. Then, the time interval is proportional to the difference in heliocentric longitudes:

$$S = \frac{TT_\oplus}{T - T_\oplus} = \frac{a^{1.5} \cdot 1}{a^{1.5} - 1} = 1.08^\text{y},$$

$$\Delta T = S \cdot \frac{\lambda}{360^\circ} = 0.17^\text{y} = 61^\text{d}.$$

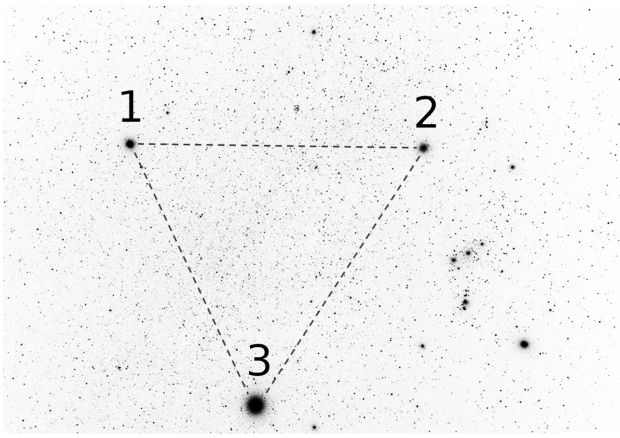
Marking Scheme:

- Question (a) Orbital radius
  1. Geometric and kinematic relations for the stationary point — **5 pt.**
  2. Valid technique to solve the equation for the orbital radius — **5 pt.**
  3. Correct numerical result — **3 pt.**
- Question (b) Time interval
  1. Synodic period, difference in heliocentric longitudes at the stationary point — **5 pt.**
  2. Correct numerical result — **2 pt.**

2 Winter Ensemble

The Winter Triangle is an asterism formed by Sirius, Procyon, and Betelgeuse. Their equatorial coordinates are listed in the table.

Star	Right Ascension	Declination
Sirius	$6^{\mathrm{h}}\,45^{\mathrm{m}}$	$-16^\circ\,45'$
Procyon	$7^{\mathrm{h}}\,40^{\mathrm{m}}$	$+05^\circ\,10'$
Betelgeuse	$5^{\mathrm{h}}\,56^{\mathrm{m}}$	$+07^\circ\,24'$



a) Write down which star is labeled by each number.

b) On what fraction of the Earth's surface are all three stars of the Winter Triangle ALWAYS above the horizon?

c) On what fraction of the Earth's surface can all three stars be above the horizon AT THE SAME TIME at least sometimes?

d) At what latitude can all three stars be at the same altitude AT THE SAME TIME?

Neglect atmospheric refraction.

Solution:

a) The stars in the figure can be identified in several ways. Betelgeuse (2) is located in the constellation of Orion, whose distinct asterism is clearly visible in the picture. Sirius, labeled number 3, is the brightest star in the night sky. Procyon is labeled number 1.

Celestial coordinates provide another method for orientation. The position of the Orion constellation is well-known — the celestial equator passes through Orion's belt. The star number 3 (Sirius) has a negative declination, as it lies noticeably lower in the sky. The remaining two stars can be distinguished by their right ascension. In the diagram, Procyon (1) is to the left of Betelgeuse (2), consistent with right ascension increasing counterclockwise when viewing the celestial sphere from the north.

b) The Winter Triangle asterism includes stars both above and below the celestial equator. For a star to be circumpolar at a given location, its declination must share the sign of the location's latitude. Since the Winter Triangle contains stars with opposing declinations, it is impossible for the entire asterism to be circumpolar simultaneously at any point on the Earth. Therefore, the fraction of the Earth's surface from which the entire Winter Triangle is circumpolar is zero.

c) To determine the latitudes from which all stars in the asterism are visible (i.e., they all rise above the horizon), we require that each star's upper culmination altitude is non-negative: $h = 90^\circ - |\varphi - \delta| \geqslant 0^\circ$. Let us find the range of latitudes that satisfies this condition for both the northernmost and southernmost stars in the group:

$$\begin{cases} 90^\circ - |\varphi - 07^\circ\,24'| \geqslant 0^\circ, \\ 90^\circ - |\varphi + 16^\circ\,45'| \geqslant 0^\circ \end{cases} \implies -82^\circ\,36' \leqslant \varphi \leqslant 73^\circ\,15'.$$

We have so far determined the range of latitudes where all stars in the asterism are theoretically visible (i.e., they each rise above the horizon at some point). However, we should verify that they can be observed simultaneously.

At the northernmost latitude, we verify whether the other two stars are above the horizon at the moment of Sirius's upper culmination. This occurs when the local sidereal time equals Sirius's right ascension: $s = \alpha = 6^\mathrm{h}\,45^\mathrm{m}$.

At the same moment the hour angles of Betelgeuse and Procyon are

$$t_B = 6^\mathrm{h}\,45^\mathrm{m} - 5^\mathrm{h}\,56^\mathrm{m} = 49^\mathrm{m} = 12.25^\circ,$$
$$t_P = 6^\mathrm{h}\,45^\mathrm{m} - 7^\mathrm{h}\,40^\mathrm{m} = -55^\mathrm{m} = -13.75^\circ.$$

We then calculate the stars' altitudes at these hour angles:

$$\sin h_B = \sin\varphi \sin\delta_B + \cos\varphi\cos\delta_B\cos t_B,$$
$$\sin h_B = \sin 73^\circ\,15' \sin 7^\circ\,24' + \cos 73^\circ\,15' \cos 7^\circ\,24' \cos 12.25^\circ,$$
$$\sin h_B = 0.40 \implies h_B = 23.7^\circ > 0^\circ;$$

$$\sin h_P = \sin\varphi \sin\delta_P + \cos\varphi\cos\delta_P\cos t_P,$$
$$\sin h_P = \sin 73^\circ\,15' \sin 5^\circ\,10' + \cos 73^\circ\,15' \cos 5^\circ\,10' \cos\left(-13.75^\circ\right),$$
$$\sin h_P = 0.37 \implies h_P = 21.4^\circ > 0^\circ.$$

At this moment, the other two stars are indeed above the horizon. As the latitude decreases from this maximum, the altitudes of all stars at culmination will initially increase. This continues until the northernmost star reaches the zenith. After this point, the culmination altitudes will begin to decrease as the stars culminate north of the zenith.

Similarly, for the southernmost latitude, we check the visibility of the other stars at the moment of Betelgeuse's upper culmination, which corresponds to the sidereal time $\alpha_B = 5^\mathrm{h}\,56^\mathrm{m}$. The hour angles of Sirius and Procyon are:

$$t_S = 5^\mathrm{h}\,56^\mathrm{m} - 6^\mathrm{h}\,45^\mathrm{m} = -49^\mathrm{m} = -12.25^\circ,$$
$$t_P = 5^\mathrm{h}\,56^\mathrm{m} - 7^\mathrm{h}\,40^\mathrm{m} = -1^\mathrm{h}\,44^\mathrm{m} = -26.00^\circ.$$

We then calculate the stars' altitudes at these hour angles:

$$\sin h_S = \sin\varphi \sin\delta_S + \cos\varphi\cos\delta_S\cos t_S,$$
$$\sin h_S = \sin(-82^\circ 36') \sin(-16^\circ 45') + \cos(-82^\circ 36')\cos(-16^\circ 45')\cos(-12.25^\circ),$$
$$\sin h_S = 0.41 \quad\Longrightarrow\quad h_S = 24^\circ > 0^\circ.$$

$$\sin h_P = \sin\varphi \sin\delta_P + \cos\varphi\cos\delta_P\cos t_P,$$
$$\sin h_P = \sin(-82^\circ 36') \sin 5^\circ 10' + \cos(-82^\circ 36')\cos 5^\circ 10'\cos(-26.00^\circ),$$
$$\sin h_P = 0.026 \quad\Longrightarrow\quad h_P = 1.5^\circ > 0^\circ.$$

At the moment of Betelgeuse's upper culmination, the other two stars are also confirmed to be above the horizon.

Therefore, the entire range of latitudes from $-82^\circ 36'$ to $73^\circ 15'$ is suitable for simultaneously observing the asterism. The fraction of the Earth's surface area within this latitudinal range is given by

$$1 - \frac{1 - \sin 73^\circ\, 15'}{2} - \frac{1 - |\sin(-82^\circ\, 36')|}{2} = 0.97.$$

d) If the altitudes of all stars in the triangle are equal, the zenith must be equidistant from the vertices of the triangle on the celestial sphere. Our goal is to find the equatorial coordinates $(\alpha_0; \delta_0)$ of this point, which is equidistant from the three stellar positions.

On the unit sphere, the condition for equal angular distances is equivalent to the dot products between the unit vector to the zenith and the unit vectors to each star being equal. Representing these vectors in the Cartesian equatorial coordinate system, we have:

$$\mathbf{r}_P = \begin{pmatrix} \cos\delta_P\cos\alpha_P \\ \cos\delta_P\sin\alpha_P \\ \sin\delta_P \end{pmatrix} = \begin{pmatrix} -0.421 \\ 0.903 \\ 0.090 \end{pmatrix}, \qquad \mathbf{r}_S = \begin{pmatrix} \cos\delta_S\cos\alpha_S \\ \cos\delta_S\sin\alpha_S \\ \sin\delta_S \end{pmatrix} = \begin{pmatrix} -0.187 \\ 0.939 \\ -0.288 \end{pmatrix},$$

$$\mathbf{r}_B = \begin{pmatrix} \cos\delta_B\cos\alpha_B \\ \cos\delta_B\sin\alpha_B \\ \sin\delta_B \end{pmatrix} = \begin{pmatrix} 0.017 \\ 0.992 \\ 0.129 \end{pmatrix}, \qquad \mathbf{r}_z = \begin{pmatrix} \cos\delta_0\cos\alpha_0 \\ \cos\delta_0\sin\alpha_0 \\ \sin\delta_0 \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix}.$$

$$\begin{cases} \mathbf{r}_P\cdot\mathbf{r}_z = \mathbf{r}_S\cdot\mathbf{r}_z\,, \\ \mathbf{r}_P\cdot\mathbf{r}_z = \mathbf{r}_B\cdot\mathbf{r}_z\,, \\ X_0^2+Y_0^2+Z_0^2 = 1 \end{cases} \Longrightarrow \begin{cases} -0.421X_0 + 0.903Y_0 + 0.090Z_0 = -0.187X_0 + 0.939Y_0 - 0.288Z_0\,, \\ -0.421X_0 + 0.903Y_0 + 0.090Z_0 = 0.017X_0 + 0.992Y_0 + 0.129Z_0\,, \\ X_0^2+Y_0^2+Z_0^2 = 1. \end{cases}$$

We can solve the system by first expressing X_0 and Y_0 in terms of Z_0 using the first two equations. Substituting these expressions into the third equation yields an equation for Z_0.

$$\begin{cases} X_0 = 6.929 Z_0\,, \\ Y_0 = -34.54 Z_0\,, \\ X_0^2 + Y_0^2 + Z_0^2 = 1 \end{cases} \quad \Longrightarrow \quad \begin{cases} X_0 = 6.929 Z_0\,, \\ Y_0 = -34.54 Z_0\,, \\ |Z_0| = |\sin\delta_0| = 0.028. \end{cases}$$

The declination of the zenith point corresponds directly to the observer's latitude. Therefore, the possible observation latitudes are $\pm\arcsin(0.028) = \pm 1.6^\circ$.

Furthermore, the geometry of the configuration indicates that for an observer in the northern hemisphere the point of equal altitude will be located near the nadir. Conversely, for an observer in the southern hemisphere it will be near the zenith.

Marking Scheme:

- Question (a) Star identification
  3 pt.: 1 pt for each correctly identified star

- Question (b) Always above the horizon
  1. Any meaningful drawing — **1 pt.**
  2. Any sound reasoning based on the drawing or celestial coordinates.
     If there is no picture, full points are still given for this part if the reasoning remains sound — **2 pt.**
  3. Correct answer (0) — **2 pt.**

- Question (c) Sometimes above the horizon
  1. Any approximately correct culminations are written down — **2 pt.**
  2. Proof of simultaneity by any correct method — **2 pt.**
  3. Correct formula for the area of the spherical segment (fraction of sphere) — **1 pt.**
  4. Correct numerical answer — **1 pt.**

- Question (d) Together at the same altitude
  1. Idea for finding the answer (understanding that it depends only on latitude and that an equidistant point needs to be found) — **2 pt.**
  2. Correct analytical (formula-based) implementation of the idea — **2 pt.**
  3. Correct numerical answer for latitude — **1 pt.**
  4. Statement about two possible answers (latitudes) — **1 pt.**

3 Swing of the Spheres

Because of annual aberration, a G2V star (Sun-like) draws an ellipse on the celestial sphere with an eccentricity of 0.4. Its annual parallax is 1 % of the maximum aberrational shift.

a) Determine the distance to the star.

b) Estimate the minimum baseline of an optical interferometer that can resolve at least some details on the stellar disk at such distance.

c) What is the apparent magnitude of such an object? Are there any known G2V stars at such distance?

d) Estimate the surface brightness of the stellar disk. Express the result in mag/arcsec^2.

e) Find the possible altitude range of the upper culmination of this star for an observer in the city of Sochi (43.6° N).

Solution:

a) The maximum aberrational displacement is determined by the Earth's full orbital velocity and is given by $\theta = v_\oplus/c = 10^{-4}$ rad $= 20.6''$. Although one could use the Earth's velocity at perihelion to account for orbital ellipticity, this refinement is not essential for this calculation.

In this case, the parallactic displacement is $0.206''$, which corresponds to a distance of

$$r = 1/0.206 \approx 4.9 \text{ [pc]}.$$

b) To resolve details on the star's disk, the telescope's angular resolution must be finer than the star's angular diameter. For a rough estimate, we can require the resolution to be better than approximately 0.2 times the angular size. Participants may use any reasonable factor for this estimation.

The angular resolution of an optical interferometer is given by:

$$\alpha \sim \frac{\lambda}{D},$$

where λ is the wavelength of observed radiation and D is the baseline (the distance between the telescopes).

The angular diameter d of the star is related to its physical radius R and distance r:

$$d = \frac{2R}{r}.$$

Equating the required resolution to the interferometer's resolution gives:

$$0.2 \cdot \frac{2R}{r} \sim \frac{\lambda}{D}.$$

Using an observation wavelength of 550 nm (visible spectrum characteristic wavelength—green light), we can solve for the required baseline D:

$$D \sim \frac{\lambda r}{0.4 R_\odot} = \frac{5.5 \cdot 10^{-7}\ \text{m} \cdot 4.9 \times 3.1 \cdot 10^{13}\ \text{km}}{0.4 \times 6.96 \cdot 10^5\ \text{km}} \sim 3 \cdot 10^2\ \text{m}.$$

This calculated distance is similar to the maximum baseline of the CHARA optical interferometer.

c) The distance to the object is relatively small, so interstellar extinction is negligible. The relation between the apparent magnitude m, absolute magnitude M, and distance r (in parsecs) is

$$m = M - 5 + 5 \log r = 4.8^\text{m} - 5 + 5 \log 4.9 \approx 3.3^\text{m}.$$

The object described would be a bright, Sun-like star. However, no stars fitting this description are observed at the calculated distance of 4.9 pc. Most nearby stars are cooler than the Sun, and the well-known Alpha Centauri system is significantly closer. The star with parameters most similar to those in this problem is τ Ceti: it is somewhat cooler (spectral type G8V) and slightly closer (3.7 pc) than the 4.9 pc distance we obtained.

d) To estimate the surface brightness of the star's disk, we neglect limb darkening and assume uniform brightness across its surface. Under this assumption, the apparent magnitude of the entire disk m can be related to the apparent magnitude per square arcsecond μ using Pogson's formula:

$$m - \mu = -2.5 \log\left(S/\text{arcsec}^2\right) = -2.5 \log\left(\pi \frac{d^2}{4}\right).$$

The angular diameter of the disk, expressed in terms of its physical diameter and distance, is converted to arcseconds as follows:

$$\mu = m - 2.5 \log\left(\pi \cdot \frac{(2R_\odot/r \cdot 206265)^2}{4}\right) = -10.6^\text{m}/\text{arcsec}^2.$$

Alternative way. It may be recalled that, in the absence of absorption, the surface brightness of objects does not depend on distance. The desired quantity is equal to the surface brightness of the solar disk. The apparent solar diameter is approximately $31' = 1\,860''$:

$$\mu = -26.7^\text{m} + 2.5 \log\left(\pi \cdot 1860^2/4\right) = -10.6^\text{m}/\text{arcsec}^2.$$

e) The eccentricity of the aberration ellipse is related to the ecliptic latitude of the star. The minor axis of the aberration ellipse is related to the semi-major axis $a = \theta$ as $b = \theta \sin \beta$, whence the eccentricity is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \sin^2 \beta} = \cos \beta.$$

Consequently, for an eccentricity of $e = 0.4$, the ecliptic latitude is $\beta = \pm \arccos 0.4 \approx \pm 66.4^\circ$.

The ecliptic longitude of the star is unspecified. Consequently, we have to estimate the possible range of altitudes at upper culmination using the known ecliptic latitude β and geographic latitude $\varphi = 43.6^\circ$. To determine the corresponding declination δ, which is required for the altitude calculation, we apply the spherical law of cosines:

$$\sin\delta = \sin\varepsilon\sin\lambda\cos\beta + \cos\varepsilon\sin\beta.$$

$$\sin\delta_{\text{min}} = \cos\varepsilon\sin\beta - \sin\varepsilon\cos\beta = \sin(\beta - \varepsilon) \quad \Longrightarrow \quad \delta_{\text{min}} = \beta - \varepsilon,$$
$$\sin\delta_{\text{max}} = \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta = \sin(\beta + \varepsilon) \quad \Longrightarrow \quad \delta_{\text{max}} = \beta + \varepsilon.$$

The minimum declination occurs when the star is located near the winter solstice point ($\sin\lambda = -1$), and the maximum declination occurs near the summer solstice point ($\sin\lambda = +1$).

Northern hemisphere	Southern hemisphere
$\beta = +66.4^\circ$	$\beta = -66.4^\circ$
$\delta_{\text{min}} = +43.0^\circ$	$\delta_{\text{min}} = -89.8^\circ$
$\delta_{\text{max}} = +89.8^\circ$	$\delta_{\text{max}} = -43.0^\circ$

We now estimate the culmination altitudes. For the northern celestial hemisphere, the observer's latitude falls within the calculated range of declinations, meaning the star's declination can equal the latitude. This implies the star can pass directly through the zenith (altitude $h = 90^\circ$) at upper culmination. To find the minimum possible altitude at upper culmination, we use the extreme declination values:

$$h(\delta_{\text{min}}) = 90^\circ - |\varphi - \delta_{\text{min}}| = 90^\circ - |43.6^\circ - 43.0^\circ| = 89.4^\circ,$$
$$h(\delta_{\text{max}}) = 90^\circ - |\varphi - \delta_{\text{max}}| = 90^\circ - |43.6^\circ - 89.8^\circ| = 43.8^\circ.$$

Thus, the altitude at upper culmination ranges from $+43.8^\circ$ to $+90^\circ$.

For the southern celestial hemisphere,

$$h(\delta_{\text{min}}) = 90^\circ - |\varphi - \delta_{\text{min}}| = 90^\circ - |43.6^\circ + 43.0^\circ| = +3.4^\circ,$$
$$h(\delta_{\text{max}}) = 90^\circ - |\varphi - \delta_{\text{max}}| = 90^\circ - |43.6^\circ + 89.8^\circ| = -43.4^\circ.$$

Thus, the altitude at upper culmination ranges from -43.4° to 3.4°.

Marking Scheme:

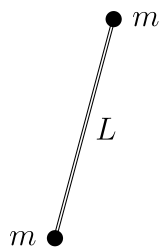
- Question (a) Distance
  1. Estimate of maximum aberration — **1 pt.**
  2. Parallax — **1 pt.**
  3. Distance from parallax — **2 pt.**

     If the distance is incorrect, subsequent items are not penalized if it is used further.

- Question (b) Interferometer baseline
  1. Angular size of the star — **2 pt.**
  2. Baseline length — **2 pt.**
- Question (c) Apparent magnitude
  1. Comparison with the Sun, using the apparent magnitude — **1 pt.**
  2. Apparent and absolute magnitude — **2 pt.**
  3. No Sun-like stars exist at such a distance — **2 pt.**
- Question (d) Surface brightness — **2 pt.**
- Question (e) Altitudes of upper culmination
  1. Ecliptic latitude from ellipse eccentricity (recalling or deriving) — **2 pt.**
  2. Range of possible declinations — **1 pt.**
  3. Altitudes of upper culmination — **2 pt.**

4 Tides on a Rod

An experimental gradiometer satellite consists of a thin, rigid, massless rod of length $L = 40$ m with small masses $m = 500$ kg at its ends. Its center moves in a circular orbit at an altitude $h = 400$ km above the Earth's surface. In equilibrium, the rod is oriented radially toward the Earth's center.



Assume that the Earth's gravitational field is the same as that of a point mass:

$$\mathbf{g}(\mathbf{r}) = -\frac{G\mathfrak{M}_\oplus}{r^3}\mathbf{r}.$$

a) Determine the orbital period of the satellite, neglecting the rod's length.

b) Find a small correction to the satellite's circular orbital speed that comes from the fact that it isn't a point, but has a finite length L:

$$v \approx \sqrt{\frac{G\mathfrak{M}_\oplus}{R_\oplus + h}}\left(1 + [\text{SOME NUMBER}]\frac{L^2}{(R_\oplus + h)^2}\right).$$

c) Find the period of small oscillations of the satellite about the equilibrium in the orbital plane, assuming that the satellite's center continues to move in a circular orbit.

Solution:

a) The radius of satellite's orbit is $r_0 = R_\oplus + h$. For a circular orbit, the centripetal acceleration is provided by gravity:

$$\frac{v_0^2}{r_0} = \frac{G\mathfrak{M}_\oplus}{r_0^2} \implies v_0 = \sqrt{\frac{G\mathfrak{M}_\oplus}{r_0}}, \quad \omega_0 = \sqrt{\frac{G\mathfrak{M}_\oplus}{r_0^3}};$$

$$T_0 = \frac{2\pi r_0}{v_0} = 2\pi\sqrt{\frac{r_0^3}{G\mathfrak{M}_\oplus}} = 2\pi\sqrt{\frac{(R_\oplus + h)^3}{G\mathfrak{M}_\oplus}} =$$

$$= 2 \times 3.14 \times \sqrt{\frac{(6.371 \cdot 10^6 + 0.4 \cdot 10^6)^3 \text{ m}^3}{6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times 5.97 \cdot 10^{24} \text{ kg}}} = 5.5 \cdot 10^3 \text{ s} = 92^\text{m} \approx 1.5^\text{h}.$$

b) Let $\ell = L/2$ be the half-length of the rod. The positions of its endpoints are $\mathbf{r}_\pm = \mathbf{r}_\text{COM} \pm \boldsymbol{\rho}$, where $|\boldsymbol{\rho}| = \ell$. By definition, their average gives the center of mass (COM):

$$\mathbf{r}_\text{COM} = \frac{\mathbf{r}_+ + \mathbf{r}_-}{2}, \qquad |r_\text{COM}| = r_0.$$

Since internal forces cancel, Newton's second law for the system gives

$$2m\mathbf{a}_\text{COM} = \mathbf{F}_+ + \mathbf{F}_- = m\mathbf{g}_+ + m\mathbf{g}_- \implies \mathbf{a}_\text{COM} = \frac{\mathbf{g}_+ + \mathbf{g}_-}{2} \implies a_\text{COM} = \frac{g_+ + g_-}{2}.$$

We now compute the gravitational acceleration at the endpoints using the binomial expansion, noting that $\ell/r_0 \ll 1$:

$$g(r_0 \pm \ell) = \frac{G\mathfrak{M}_\oplus}{(r_0 \pm \ell)^2} = \frac{G\mathfrak{M}_\oplus}{r_0^2}\left(1 \pm \frac{\ell}{r_0}\right)^{-2} = \frac{G\mathfrak{M}_\oplus}{r_0^2}\left[1 \mp 2\frac{\ell}{r_0} + 3\frac{\ell^2}{r_0^2} + \cdots\right].$$

"Averaging" the two values cancels the odd term:

$$a_{\text{COM}} = \frac{g(r_0+\ell) + g(r_0-\ell)}{2} \approx \frac{G\mathfrak{M}_\oplus}{r_0^2}\left(1 + 3\frac{\ell^2}{r_0^2}\right).$$

Alternative way. Let us denote $x = \ell/r_0 \ll 1$. We start by simplifying the expression:

$$\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} = \frac{(1-x)^2 + (1+x)^2}{(1+x)^2(1-x)^2} = 2\cdot\frac{1+x^2}{(1-x^2)^2} = 2\cdot\frac{1+x^2}{1-2x^2+x^4} =$$

$$= 2\cdot\frac{(1+x^2)(1+2x^2)}{(1-2x^2+x^4)(1+2x^2)} = 2\cdot\frac{\boxed{1+3x^2} + \underline{2x^4}}{\boxed{1} + (2x^2-3)\underline{x^4}} \overset{\text{up to } x^2}{\approx} 2\cdot(1+3x^2).$$

This result simplifies the calculation of the center-of-mass acceleration:

$$a_{\text{COM}} = \frac{1}{2}(g_+ + g_-) = G\mathfrak{M}_\oplus \cdot \frac{1}{2}\left[\frac{1}{(r_0+\ell)^2} + \frac{1}{(r_0-\ell)^2}\right] =$$

$$= \frac{G\mathfrak{M}_\oplus}{r_0^2}\cdot\frac{1}{2}\left[\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2}\right] \approx \frac{G\mathfrak{M}_\oplus}{r_0^2}\left(1 + 3\frac{\ell^2}{r_0^2}\right).$$

For a circular COM orbit,

$$a_{\text{COM}} = \frac{v^2}{r_0} \implies v^2 \approx \frac{G\mathfrak{M}_\oplus}{r_0}\left(1 + 3\frac{\ell^2}{r_0^2}\right) \implies v \approx \sqrt{\frac{G\mathfrak{M}_\oplus}{r_0}}\left(1 + \frac{3}{8}\frac{L^2}{(R_\oplus + h)^2}\right).$$

Thus, [SOME NUMBER] $= \dfrac{3}{8}$, and the relative correction to the velocity is

$$\frac{v - v_0}{v_0} = \frac{3}{8}\left(\frac{L}{R_\oplus + h}\right)^2 = \frac{3}{8}\times\left(\frac{40\ \text{m}}{6.371\cdot 10^6\ \text{m} + 0.4\cdot 10^6\ \text{m}}\right)^2 = 1.3\cdot 10^{-11}.$$

c) When the rod is tilted by a small angle θ, each end feels a force that is

- slightly stronger/weaker because of being nearer/farther, and
- points in a direction turned by a small angle because the end is shifted sideways.

The combination of these effects produces a net torque about the center of mass, which acts as a restoring torque:

$$M = F_+ \cdot \frac{L}{2}\sin\theta_+ - F_- \cdot \frac{L}{2}\sin\theta_-,$$

where

$$F_\pm = m \cdot g(r_0 \pm \ell \cos\theta),$$

$$\theta_\pm \approx \theta \mp \frac{\ell \sin\theta}{r_0}.$$

We begin by expanding the gravitational acceleration at each end of the rod $g(r_0 \pm \ell\cos\theta)$, noting that $\ell/r_0 \ll 1$:

$$g(r_0 \pm \ell\cos\theta) = \frac{G\mathfrak{M}_\oplus}{(r_0 \pm \ell\cos\theta)^2} = \frac{G\mathfrak{M}_\oplus}{r_0^2}\left(1 \pm \frac{\ell\cos\theta}{r_0}\right)^{-2} = \frac{G\mathfrak{M}_\oplus}{r_0^2}\left[1 \mp 2\frac{\ell\cos\theta}{r_0} + \cdots\right].$$

For small angles $\theta \ll 1$, we use the approximations $\sin\theta \approx \theta,\ \cos\theta \approx 1$. The restoring torque about the center of mass, calculated to first order, is

$$M \approx \frac{G\mathfrak{M}_\oplus m}{r_0^2} \cdot \ell\theta \cdot \left[\left(1 - 2\frac{\ell}{r_0}\right) \cdot \left(1 - \frac{\ell}{r_0}\right) - \left(1 + 2\frac{\ell}{r_0}\right) \cdot \left(1 + \frac{\ell}{r_0}\right)\right] \approx -\frac{6G\mathfrak{M}_\oplus m}{r_0^2}\frac{\ell^2}{r_0}\theta.$$

The moment of inertia of the two end masses about the COM in the orbital plane is

$$J = 2m\ell^2.$$

Using the torque above, we obtain the equation of motion for a harmonic oscillator

$$J\ddot{\theta} = M \qquad \Longrightarrow \qquad \ddot{\theta} + 3\frac{G\mathfrak{M}_\oplus}{r_0^3}\,\theta = 0.$$

Therefore, the angular frequency and period of small librations are:

$$\omega_{\text{osc}} = \sqrt{3} \cdot \sqrt{\frac{G\mathfrak{M}_\oplus}{r_0^3}} = \sqrt{3}\omega_0,$$

$$T_{\text{osc}} = \frac{2\pi}{\omega_{\text{osc}}} = \frac{T_0}{\sqrt{3}} = \frac{5.5 \cdot 10^3 \text{ s}}{1.73} = 3.2 \cdot 10^3 \text{ s} = 53^{\text{m}} \approx 0.9^{\text{h}}.$$

Elegant shortcut. In the reference frame co-rotating with the rod's center of mass, the effective potential combines Earth's gravity and the centrifugal potential:

$$U(r) = -\frac{G\mathfrak{M}_\oplus}{r} - \frac{1}{2}\omega_0^2 r^2.$$

The center of mass is in mechanical equilibrium at the orbital radius $r = r_0$, satisfying:

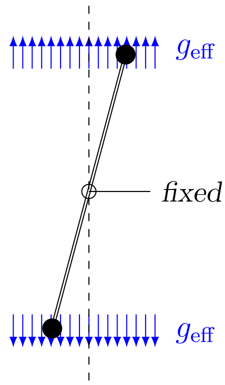
$$U'(r)\Big|_{r=r_0} = 0 = \frac{G\mathfrak{M}_\oplus}{r_0^2} - \omega_0^2 r_0 \qquad \Longrightarrow \qquad \omega_0^2 = \frac{G\mathfrak{M}_\oplus}{r_0^3},$$

which is the standard expression for the orbital angular velocity.

The tidal acceleration near r_0 is given by the second derivative of the potential. To first order:

$$a_{\text{tidal}}(r) \approx U''(r_0) \cdot (r - r_0) = \left(-\frac{2G\mathfrak{M}_\oplus}{r_0^3} - \omega_0^2 \right) \cdot (r - r_0) = -\frac{3G\mathfrak{M}_\oplus}{r_0^3} \cdot (r - r_0).$$

For small angular deflections, the system can be modeled as two simple pendulums of length $L/2$, oscillating in the effective tidal field, each in a uniform field with acceleration



$$g_{\text{eff}} = \frac{3G\mathfrak{M}_\oplus}{r_0^3} \cdot \frac{L}{2}.$$

The corresponding oscillation period is

$$T_{\text{osc}} = 2\pi\sqrt{\frac{L/2}{g_{\text{eff}}}} = 2\pi\sqrt{\frac{r_0^3}{3G\mathfrak{M}_\oplus}} = \frac{T_0}{\sqrt{3}}.$$

Marking Scheme:

- Question (a) Orbital period
  - Correct expression for the period — **2 pt.**
  - Calculations and correct answer — **2 pt.**
- Question (b) Correction to circular speed
  - Correct Newton's 2nd law — **2 pt.**
  - Correct series expansion — **4 pt.**
  - Correct numerical result — **2 pt.**
- Question (c) Period of small oscillations
  - Correct physical laws and relations that can be used to derive the differential equation for harmonic oscillations — **4 pt.**
  - Correct harmonic motion equation — **2 pt.**
  - Correct numerical result — **2 pt.**

5 Above the Photosphere

For a main-sequence star, the difference in free-fall acceleration in the photosphere and at the upper boundary of the chromosphere (10 000 km above the photosphere), measured above the star's pole, is 8 m/s^2. It is also known that from the upper boundary of the chromosphere, only 0.7 % of the entire photosphere is visible.

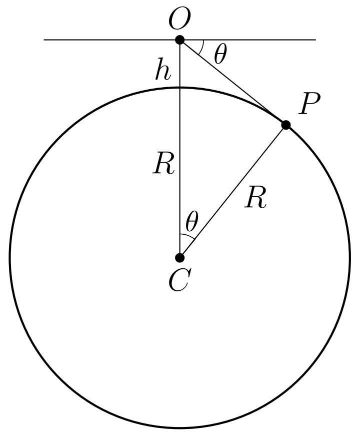
a) Estimate the mass, radius, and temperature of the star.

b) Which spectral and luminosity class does this star belong to?

c) Estimate the minimum possible rotational period of the star.

Solution:

a) An observer (point O) at a height h above the photosphere of a star of radius R sees a spherical cap defined by the angle θ. The solid angle subtended by this cap is



$$\Omega = 2\pi(1-\cos\theta), \quad \text{where } \cos\theta = \frac{R}{R+h}.$$

The fraction f of the total 4π steradians is therefore:

$$f = \frac{\Omega}{4\pi} = \frac{2\pi(1-\cos\theta)}{4\pi} = \frac{1-\cos\theta}{2} = \frac{1-\frac{R}{R+h}}{2} = \frac{h}{2(R+h)}.$$

Solving for the radius R:

$$R = \frac{h(1-2f)}{2f} = \frac{1.0\cdot 10^7\ \text{m} \times (1-2\times 0.007)}{2\times 0.007} = \frac{1.0\cdot 10^7\ \text{m}\times 0.986}{0.014} \approx 7.0\cdot 10^8\ \text{m} = 1.0\,R_\odot.$$

The free-fall acceleration at a distance r from the center:

$$g = \frac{G\mathfrak{M}}{r^2}.$$

The difference in the gravitational acceleration between two locations is

$$\Delta g = \frac{G\mathfrak{M}}{R^2} - \frac{G\mathfrak{M}}{(R+h)^2} = G\mathfrak{M}\left(\frac{1}{R^2} - \frac{1}{(R+h)^2}\right).$$

Given $h \ll R$, we simplify the expression in parentheses:

$$\frac{1}{R^2} - \frac{1}{(R+h)^2} = \frac{(R+h)^2 - R^2}{R^2(R+h)^2} = \frac{R^2+2Rh+h^2-R^2}{R^2(R+h)^2} = \frac{2Rh+h^2}{R^2(R+h)^2} \approx \frac{2Rh}{R^2\cdot R^2} = \frac{2h}{R^3}.$$

Thus,

$$\Delta g \approx \frac{2G\mathfrak{M}h}{R^3}.$$

Solving for the mass:

$$\mathfrak{M} \approx \frac{\Delta g\, R^3}{2Gh} = \frac{8\ \text{m/s}^2 \times (7.0 \cdot 10^8\ \text{m})^3}{2 \times 6.67 \cdot 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times 1.0 \cdot 10^7\ \text{m}} \approx 2.06 \cdot 10^{30}\ \text{kg} = 1.04\,\mathfrak{M}_\odot.$$

Using the mass-luminosity relation $L \propto \mathfrak{M}^4$ for main-sequence stars, we find the luminosity relative to the Sun:

$$\frac{L}{L_\odot} \approx \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^4 \approx 1.04^4 \approx 1.2.$$

The Stefan-Boltzmann law $L = 4\pi R^2 \sigma T^4$ in relative terms is

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4 \quad \Longrightarrow \quad \frac{T}{T_\odot} = \sqrt[4]{\frac{L/L_\odot}{(R/R_\odot)^2}} = \sqrt[4]{\frac{1.04^4}{1.0^2}} \approx 1.04.$$

b) Such temperature is typical for early G-type or late F-type stars. They are quite similar to the Sun (G2 V, 5778 K). The star is on the main sequence (luminosity class V).

c) The minimum rotational period occurs when the star's equatorial velocity reaches the first cosmic (circular) velocity. At this limit, the centrifugal force balances gravity at the equator. Hence, the corresponding minimum period is the orbital period:

$$T_{\text{min}} = \frac{2\pi R}{v_{\text{max}}} = 2\pi\sqrt{\frac{R^3}{G\mathfrak{M}}} = 2 \times 3.14 \times \sqrt{\frac{(7 \cdot 10^8\ \text{m})^3}{6.67 \cdot 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times 2 \cdot 10^{30}\ \text{kg}}} \approx 1.0 \cdot 10^4\ \text{s}.$$

It is important to note that this is a simplified estimate. In reality, as rotational velocity increases, the star deforms into an oblate spheroid. A more accurate treatment would require modeling the surface as an equipotential, where the effective gravity incorporates the centrifugal potential. However, for the purpose of a rough estimation, the spherical model is adequate.

Marking Scheme:

- Question (a) Mass, radius, and temperature
  1. Estimate for the radius — **1+1 pt.**
  2. Estimate for the mass — **2+2 pt.**
  3. Estimate for the luminosity — **2+2 pt.**
  4. Estimate for the temperature — **2+2 pt.**
- Question (b) Spectral type — **2 pt.**
- Question (c) Minimum rotational period
  - Maximum possible velocity on the equator or equivalent — **2 pt.**
  - Minimum rotational period — **2 pt.**

6 Glimpse of GLIMPSE

The cluster GLIMPSE-C01 is located in the Milky Way. The tip of the red giant branch (TRGB) corresponds to an apparent magnitude of $K = 8.7^{\mathrm{m}}$. Its absolute magnitude, assuming a metallicity of [Fe/H] = −1.5, is $M_K = -6.1^{\mathrm{m}}$ (Ivanov et al., 2005).

a) Assuming the extinction is $a_K = 0.45$ mag/kpc, estimate the distance to the cluster.

b) Estimate the total extinction A_V in the V band, given the cluster's Galactic coordinates: latitude $b = 0°$, longitude $l = 31°$.
NOTE. According to Rieke & Lebofsky (1985), $a_V/a_K = 9$.

c) In reality, the extinction, even in the K band, is uncertain: $a_K = 0.45 \pm 0.08$ mag/kpc. Find the uncertainty in the distance.

d) Is this cluster more likely a globular cluster or an open cluster? Explain.

Solution:

a) The relationship between the apparent magnitude, absolute magnitude, distance, and interstellar extinction is given by

$$K = M_K - 5 + 5\lg(r/\mathrm{pc}) + a_K \cdot (r/1000\ \mathrm{pc}) \quad \Longrightarrow \quad 5\lg(r/\mathrm{pc}) + 0.45 \cdot (r/1000\ \mathrm{pc}) = 19.8^{\mathrm{m}}.$$

We solve this equation numerically using the bisection method. Since both terms on the left-hand side are monotonically increasing functions of r, the equation has a single root. Define the function:

$$f(r) = 5\lg(r/\mathrm{pc}) + 0.45 \cdot (r/1000\ \mathrm{pc}) - 19.8.$$

We evaluate this function at several test points to bracket the root:

r/pc	[1000	10000]	[3000	5000]	4000
$f(r)$	−4.35	4.70	−1.06	0.94	0.01

Therefore, the distance is approximately 4 kpc.

b) The problem provides the ratio of absorption coefficients in two bands. Given the previously determined distance and the absorption coefficient in the K band, the total absorption in the V band is:

$$A_V = a_V \cdot r = 9a_K \cdot r = 9 \times 0.45\ \mathrm{mag/kpc} \times 4\ \mathrm{kpc} = 16.2\ \mathrm{mag}.$$

c) The uncertainty in the extinction value is significant. Although this uncertainty is symmetric in magnitude space, it translates into an asymmetric range of possible distances. The most straightforward method to determine this range is to calculate the distance using the upper and lower bounds of the extinction coefficient:

$$5\lg(r_{\min}/\text{pc}) + 0.53 \cdot (r_{\min}/1000\ \text{pc}) = 19.8^{\text{m}} \quad \Longrightarrow \quad r_{\min} = 3.7 \cdot 10^3\ \text{pc}.$$
$$5\lg(r_{\max}/\text{pc}) + 0.37 \cdot (r_{\max}/1000\ \text{pc}) = 19.8^{\text{m}} \quad \Longrightarrow \quad r_{\max} = 4.3 \cdot 10^3\ \text{pc}.$$

Thus, accounting for rounding, the distance estimate is 4.0 ± 0.3 kpc. In reality, a more precise calculation using better initial data would likely yield a slightly larger upper uncertainty, revealing an asymmetric error interval.

d) The specified condition of low metallicity immediately rules out a young open cluster. The high absolute magnitude of the tip of the red giant branch indicates the presence of highly luminous, evolved stars. This stellar population is more typical for a globular cluster or a very old open cluster — the latter being quite rare. The object's location in the Galactic plane, combined with its longitude pointing toward the central region, is consistent with the known distribution of globular clusters, which can have such coordinates.

In conclusion, the object is most likely a globular cluster (which is consistent with its actual identity), although an old open cluster remains a remote theoretical possibility.

Marking Scheme:

- Question (a) Distance to the cluster
  1. Equation relating stellar magnitudes, distance, and absorption — **3 pt.**
  2. Number of possible solutions to the equation — **2 pt.**
  3. Correct answer — **2 pt.**
- Question (b) Total extinction in V band
  1. Relation for absorption in two bands — **2 pt.**
  2. Correct answer — **2 pt.**
- Question (c) Uncertainty in the distance
  1. Relation for the uncertainty in distance or stating the method of calculating the boundary values — **2 pt.**
  2. Correct answers — **1+1 pt.**
- Question (d) Cluster type
  1. Correct reasoning concerning the cluster's classification — **4 pt.**
  2. Correct binary answer — **1 pt.**

7 Nothing to Do There

A spacecraft moves along a Hohmann transfer ellipse in the ecliptic plane from the Earth to Jupiter. This ellipse touches the Earth's orbit at perihelion and Jupiter's orbit at aphelion.

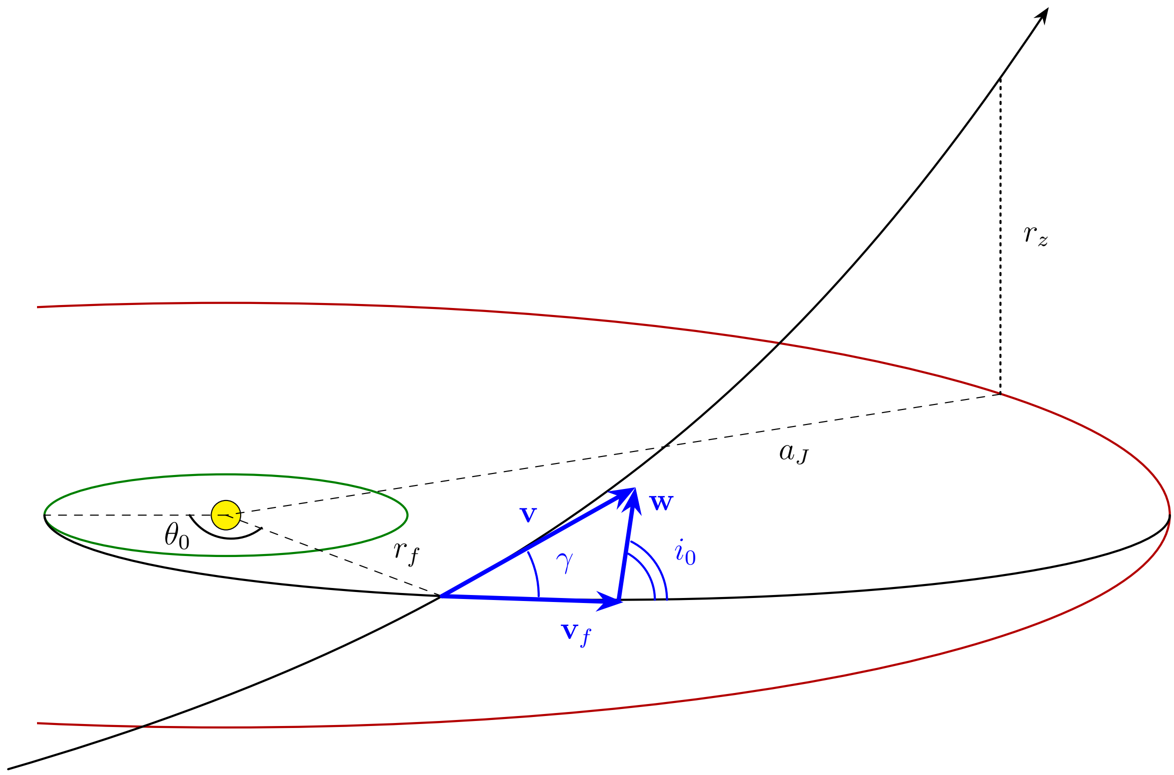
a) How long would the spacecraft take to travel from the Earth to Jupiter along this trajectory?

At true anomaly $\theta_0 = 90^\circ$ (heliocentric distance r_f), the engine is fired briefly. The spacecraft receives a velocity change $\mathbf{w}$ with magnitude $|\mathbf{w}| = \frac{1}{2}|\mathbf{v}_f|$, where $\mathbf{v}_f$ is its current velocity. The impulse makes an angle $i_0 = 20^\circ$ to the plane of the initial orbit, and its projection onto the ecliptic is aligned with the initial velocity. Assume the Earth's and Jupiter's orbits are circular and lie in the same plane.

b) By what angle γ does the spacecraft's velocity vector turn after the impulse?

c) Find the semi-major axis, eccentricity, and inclination of the new orbit after the impulse.

d) How far above Jupiter's orbit (r_z) will the spacecraft fly by?



Solution:

a) First, we determine the semi-major axis and eccentricity of the Hohmann transfer ellipse:

$$a_0 = \frac{a_\oplus + a_J}{2} = \frac{1.00 \text{ au} + 5.20 \text{ au}}{2} = 3.10 \text{ au},$$
$$e_0 = 1 - \frac{a_\oplus}{a_0} = 1 - \frac{1.00}{3.10} = 0.68.$$

Let us estimate the travel time. The transfer along the Hohmann ellipse covers half its circumference, so the time of flight is half its orbital period:

$$\Delta T/\text{yr} = \frac{1}{2}(a_0/\text{au})^{3/2} = \frac{1}{2} \times 3.10^{3/2} \approx 2.70.$$

b) The distance from the Sun on an elliptical orbit depends on the true anomaly as follows:

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos\theta}.$$

For $\theta_0 = 90^\circ$ we obtain

$$r_f = a_0\left(1 - e_0^2\right) = 3.10 \text{ au} \times (1 - 0.68^2) = 1.68 \text{ au}.$$

We can then find the velocity at this point using the vis-viva equation:

$$v_f^2 = G\mathfrak{M}_\odot\left(\frac{2}{r_f} - \frac{1}{a_0}\right)$$
$$\implies \quad v_f = \sqrt{6.67\cdot 10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times \frac{1.99\cdot 10^{30} \text{ kg}}{1.496\cdot 10^{11} \text{ m}} \times \left(\frac{2}{1.68} - \frac{1}{3.10}\right)} = 27.8 \text{ km/s}.$$

The magnitude of the velocity increment is $|\mathbf{w}| = w = 0.5v_f = 13.9$ km/s. The spacecraft's final velocity magnitude v after the impulse can be found by applying the law of cosines to the velocity vector triangle:

$$v^2 = v_f^2 + w^2 - 2v_f\cdot w\cdot\cos(180^\circ - i_0)$$
$$\implies \quad v/\tfrac{\text{km}}{\text{s}} = \sqrt{27.8^2 + 13.9^2 + 2\times 27.8\times 13.9\times\cos 20^\circ} = 41.2.$$

Finally, we determine the new orbital parameters. The inclination angle γ of the velocity vector relative to the ecliptic plane is found by applying the law of cosines again:

$$w^2 = v_f^2 + v^2 - 2vv_f\cos\gamma$$
$$\implies \quad \gamma = \arccos\frac{v_f^2 + v^2 - w^2}{2vv_f} = \arccos\frac{27.8^2 + 41.2^2 - 13.9^2}{2\times 41.2\times 27.8} = 6.6^\circ.$$

c) The new orbit's semi-major axis is found using the vis-viva equation:

$$v^2 = G\mathfrak{M}_\odot\left(\frac{2}{r_f} - \frac{1}{a}\right)$$

$$\implies \quad a = \frac{r_f}{2 - \dfrac{v^2 r_f}{G\mathfrak{M}_\odot}} = \frac{1.68\ \text{au}}{2 - \dfrac{(41.2\cdot 10^3\ \text{m/s})^2 \times 1.68 \times 1.496\cdot 10^{11}\ \text{m}}{6.67\cdot 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times 1.99\cdot 10^{30}\ \text{kg}}} = -1.4\ \text{au}.$$

The orbit is hyperbolic. We estimate the orbit's eccentricity using the conservation of angular momentum. This requires finding the angle between the new velocity vector and the radius vector.

We introduce a Cartesian coordinate system with its origin at the Sun. The x-axis is directed toward the vertex of the focal parameter, the y-axis toward the apocenter of the Hohmann orbit, and the z-axis is perpendicular to the ecliptic. The angle between $\mathbf{r}_f$ and $\mathbf{v}_f$ is then determined from the law of conservation of angular momentum:

$$v_f r_f \sin\alpha = \sqrt{G\mathfrak{M}_\odot a_0(1-e_0^2)} = \sqrt{G\mathfrak{M}_\odot r_f} \quad \implies \quad \alpha = 55.9^\circ.$$

The components of the velocity vector $\mathbf{v}_f$ are

$$\begin{aligned} v_{f,x} &= v_f\cos\alpha = 15.6\ \text{km/s},\\ v_{f,y} &= v_f\sin\alpha = 23.0\ \text{km/s},\\ v_{f,z} &= 0\ \text{km/s}. \end{aligned}$$

The new velocity vector $\mathbf{v}$, which is inclined to the ecliptic, has a projection onto the ecliptic plane that also forms an angle α with the radius vector. Its components are

$$\begin{aligned} v_x &= v\cos\alpha\cos\gamma = 22.9\ \text{km/s},\\ v_y &= v\sin\alpha\cos\gamma = 33.9\ \text{km/s},\\ v_z &= v\sin\gamma = 4.8\ \text{km/s}. \end{aligned}$$

The position vector $\mathbf{r}_f$ has components:

$$\begin{aligned} r_{f,x} &= r_f = 1.68\ \text{au},\\ r_{f,y} &= 0\ \text{au},\\ r_{f,z} &= 0\ \text{au}. \end{aligned}$$

The angle β between $\mathbf{r}_f$ and $\mathbf{v}$ is found from the dot product:

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{r}_f}{v\cdot r_f} = \frac{v_x\cdot r_{f,x} + v_y\cdot r_{f,y} + v_z\cdot r_{f,z}}{v\cdot r_f} = 0.557 \quad \implies \quad \beta = 56.1^\circ.$$

The eccentricity of the new orbit is calculated using the conservation of angular momentum:

$$v r_f \sin\beta = \sqrt{G\mathfrak{M}_\odot\, a\,(1-e^2)}$$

$$\implies \quad e = \sqrt{1 - \frac{v^2 r_f^2 \sin^2\beta}{G\mathfrak{M}_\odot a}} =$$

$$= \sqrt{1 - \frac{(41.2\cdot 10^3\ \text{m/s})^2 \times (1.68\times 1.496\cdot 10^{11}\ \text{m})^2 \times \sin^2 56.1^\circ}{6.67\cdot 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \times 1.99\cdot 10^{30}\ \text{kg} \times (-1.4\times 1.496\cdot 10^{11}\ \text{m})}} = 1.9.$$

The orbital inclination is determined from the z-component of the angular momentum vector:

$$\mathbf{L} = \mathbf{r}\times\mathbf{v} = \begin{pmatrix} r_y v_z - r_z v_y \\ r_z v_x - r_x v_z \\ r_x v_y - r_y v_x \end{pmatrix}.$$

The inclination of the new orbital plane relative to the ecliptic is given by

$$\frac{L_z}{L} = \frac{r_{f,x} v_y - r_{f,y} v_x}{\sqrt{(r_{f,y}v_z - r_{f,z}v_y)^2 + (r_{f,z}v_x - r_{f,x}v_z)^2 + (r_{f,x}v_y - r_{f,y}v_x)^2}} = 0.990,$$

$$\therefore\ i = 90^\circ - \arcsin\frac{L_z}{L} = 90^\circ - \arcsin 0.990 = 8^\circ.$$

d) To determine the point where the spacecraft will pass above Jupiter's orbit, we use the condition that its position in the previously defined coordinate system must satisfy $r_x^2 + r_y^2 = a_J^2$. Calculating this position accurately requires finding the rotation angle of the line of apsides relative to the radius vector at the maneuver point (the orbit's ascending node). We begin with finding the true anomaly θ_1 of the maneuver point on the hyperbolic trajectory:

$$r_f = \frac{a(1-e^2)}{1+e\cos\theta_1}$$

$$\implies \quad \cos\theta_1 = \frac{1}{e}\left[\frac{a(1-e^2)}{r_f} - 1\right] = \frac{1}{1.9}\left[\frac{(-1.4)\times(1-1.9^2)}{1.68} - 1\right] = 0.62 \quad \implies \quad \theta_1 = 51^\circ.$$

Let r be the distance from the Sun to a point on the trajectory. The Cartesian coordinates of the point are defined as:

$$\begin{aligned} r_x &= r\cos(\theta-\theta_1), \\ r_y &= r\sin(\theta-\theta_1)\cos i, \\ r_z &= r\sin(\theta-\theta_1)\sin i. \end{aligned}$$

The condition for passing above Jupiter's orbit leads to the following equation after substituting the orbital equation:

$$(r\cos(\theta-\theta_1))^2 + (r\sin(\theta-\theta_1)\cos i)^2 = a_J^2,$$

$$\frac{1}{(1+e\cos\theta)^2}\cdot\left(\cos^2(\theta-\theta_1)+\sin^2(\theta-\theta_1)\cos^2 i\right) = \left(\frac{a_J}{a(1-e^2)}\right)^2.$$

This equation is solved for the true anomaly θ using the bisection method. We define the function

$$f(\theta) = \frac{1}{(1+e\cos\theta)^2}\cdot\left(\cos^2(\theta-\theta_1)+\sin^2(\theta-\theta_1)\cos^2 i\right) - \left(\frac{a_J}{a(1-e^2)}\right)^2.$$

The function values at several test points are listed in the table below.

θ	[10°	100°]	[75°	90°]	[95°	99°]	98°
$f(\theta)$	−1.86	0.227	−1.53	−0.99	−0.56	0.03	−0.15

The root of the equation lies between $\theta = 98°$ and $\theta = 99°$. For an approximate solution, we take $\theta = 99°$. The corresponding z-coordinate is then

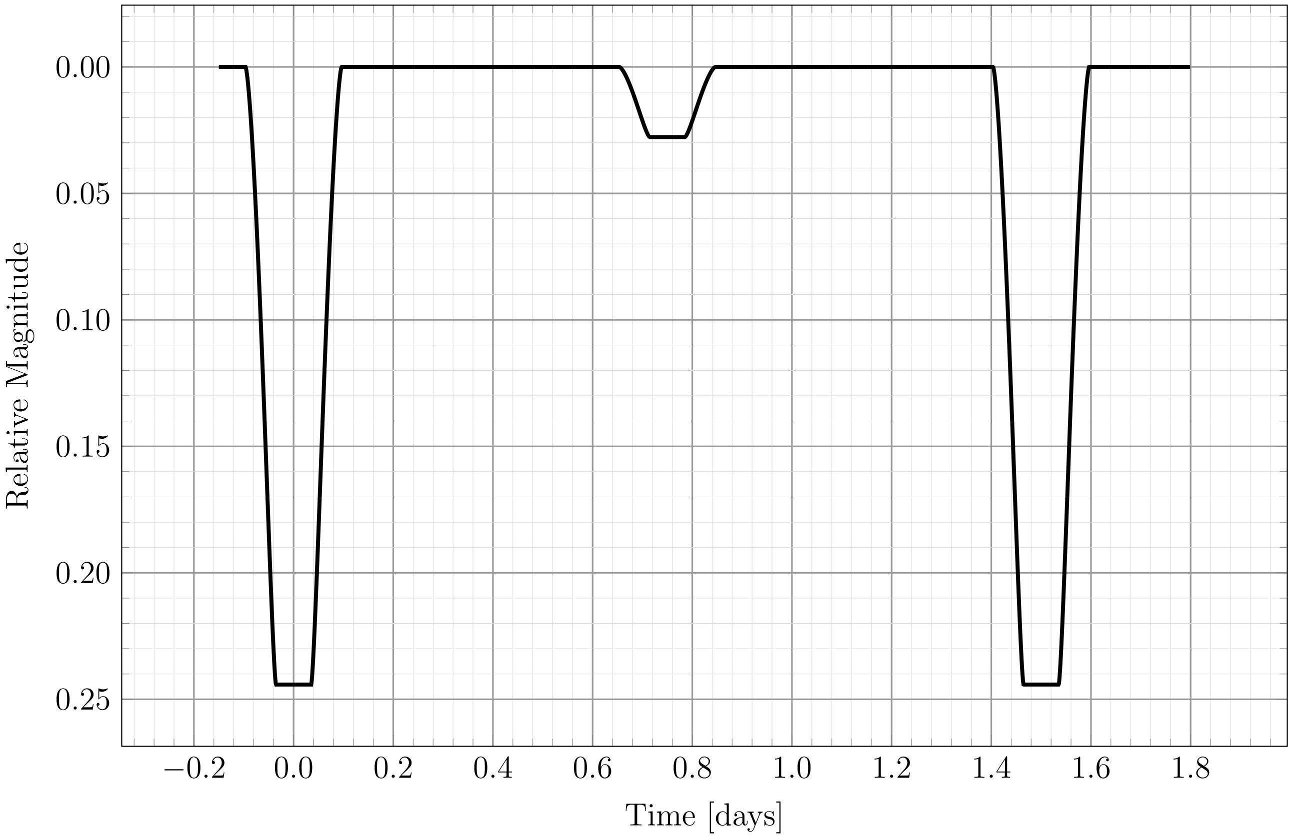
$$r_z = \frac{a(1-e^2)}{1+e\cos\theta}\sin(\theta-\theta_1)\sin i \approx 0.54 \text{ au}.$$

Marking Scheme:

- Question (a) Hohmann transfer time
  1. Semi-major axis and eccentricity of the ellipse — **1+1 pt. = 2 pt.**
  2. Transfer time (Kepler's law, half time, and correct result) —**1+1+1 pt. = 3 pt.**
- Question (b) Velocity vector turn
  1. Formula for the elliptical radius vector, substituting $\theta_0 = 90°$ — **1 pt.**
  2. Velocity at the impulse point using the vis-viva equation — **1 pt.**
  3. Using the law of cosines to find the velocity v, angle γ, and correct result — **1+1+1 pt. = 3 pt.**
- Question (c) Parameters of new orbit
  1. Correct estimate for the semi-major axis — **2 pt.**
  2. Angle α on the initial orbit and projections of $\mathbf{r}_f$ and $\mathbf{v}_f$ — **1 pt.**
  3. Expressions for the angle β and for the magnitude of angular momentum — **1 pt.**
  4. Correct estimate for the eccentricity — **2 pt.**
  5. Correct estimate for the inclination — **2 pt.**
- Question (d) Above Jupiter's orbit — **2 pt.**

8 Stellar Blends

The figure shows the light curve of an eclipsing binary system observed in the V band. The eclipses in the system are central. Both components are main-sequence stars, and one has spectral type A0 V.



a) Determine the spectral type of the second component of the system.

b) Estimate the distance between the two components of the system.

c) What is wrong with the given light curve?

d) Sketch the curve of the system's color index $(B - V)$ as a function of orbital phase, still assuming the validity of the given light curve.

Solution:

a) The figure shows the V-band light curve of a central-eclipse binary. One component is A0 V with $T_{\rm A0} = 9700$ K. The two eclipse depths read from the curve are

$$\Delta m_{\rm pri} = 0.244 \text{ mag} \qquad \text{(deeper, hotter star eclipsed);}$$
$$\Delta m_{\rm sec} = 0.028 \text{ mag.}$$

The equal spacing between successive minima, together with their equal widths, indicates that the relative orbit is circular.

A magnitude drop Δm corresponds to a fractional flux loss

$$\delta \equiv 1 - 10^{-0.4\,\Delta m}.$$

Hence

$$\delta_{\text{pri}} = 1 - 10^{-0.4\times 0.244} = 0.2013,$$
$$\delta_{\text{sec}} = 1 - 10^{-0.4\times 0.028} = 0.0255.$$

For a central eclipse in which the smaller (cooler) star passes in front, the observed flux loss at primary minimum corresponds to the V-band flux from the portion of the hotter star hidden by the cooler companion, while at secondary minimum it corresponds to the entire flux of the cooler star being obscured:

$$\Delta F_{\text{pri}} = I_{h,V}\,A_c, \qquad \Delta F_{\text{sec}} = I_{c,V}\,A_c,$$

with A_c denoting the projected disc area of the cool star and $I_{h,V}, I_{c,V}$ the V-band surface brightnesses of the hot and cool stars, respectively. Taking the ratio of the two light losses (or equivalently, of the corresponding fractional losses) gives

$$k_V \equiv \frac{I_{h,V}}{I_{c,V}} = \frac{\Delta F_{\text{pri}}}{\Delta F_{\text{sec}}} = \frac{\delta_{\text{pri}}}{\delta_{\text{sec}}} = \frac{0.2013}{0.0255} = 7.9.$$

The secondary depth also fixes the projected area ratio:

$$\delta_{\text{sec}} = \frac{F_{c,V}}{F_{h,V} + F_{c,V}} = \frac{I_{c,V}\,A_c}{I_{h,V}\,A_h + I_{c,V}\,A_c} = \frac{A_c/A_h}{k_V + A_c/A_h}$$
$$\implies \quad \frac{A_c}{A_h} = \frac{k_V\,\delta_{\text{sec}}}{1-\delta_{\text{sec}}} = \frac{7.9\times 0.0255}{1-0.0255} = 0.207,$$

so that the radius ratio is

$$\frac{R_c}{R_h} = \sqrt{\frac{A_c}{A_h}} = \sqrt{0.207} = 0.455.$$

Now we consider A0 V as the hotter, larger component.

Since bolometric corrections are negligible for A–G type dwarfs, we may take $I_{\text{bol}} \propto T^4$ and approximate the V-band surface-brightness ratio by

$$\frac{T_h}{T_c} \approx k_V^{1/4} = 7.9^{1/4} = 1.68.$$

With $T_h = T_{\text{A0}} = 9700$ K, the companion's temperature follows as $T_c = T_h/1.68 \approx 5.77\times 10^3$ K, which corresponds to a Sun-like G2 V star. The radius ratio derived from the light-curve minima is consistent with $R_{\text{G2}} = 1.0\,R_\odot$ and $R_{\text{A0}} = 2.2\,R_\odot$.

b) Adopt main-sequence masses $M_{\mathrm{A0\,V}} \approx 2.2\,\mathfrak{M}_\odot$ and $\mathfrak{M}_{\mathrm{G2\,V}} = 1.0\,\mathfrak{M}_\odot$. From the light curve we read the orbital period $P = 1.5$ d.

Hence Kepler's third law in (au; yr; $\mathfrak{M}_\odot$) units gives

$$a/\mathrm{au} = \sqrt[3]{(\mathfrak{M}_1 + \mathfrak{M}_2) \cdot (P/\mathrm{yr})^2} = \left[3.2 \times \left(\frac{1.5}{365.25}\right)^2\right]^{1/3} = 0.038;$$

$$a = 0.038\ \mathrm{au} = 5.7 \cdot 10^6\ \mathrm{km} \approx 8\,R_\odot.$$

c) Note that $R_{\mathrm{A0\,V}} + R_{\mathrm{G2\,V}} \approx 2.2\,R_\odot + 1.0\,R_\odot = 3.2\,R_\odot$.

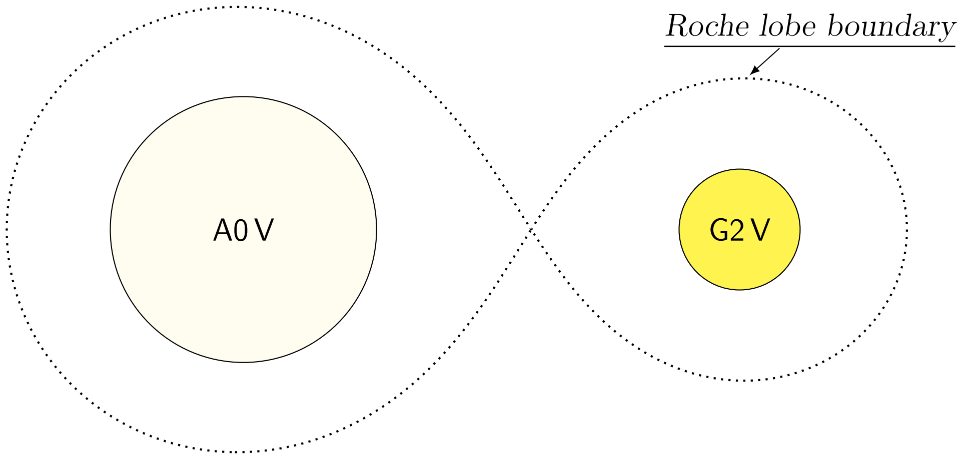


Figure 2: Schematic view of the system

Because the stars are so close, out-of-eclipse variability should arise from both *ellipsoidal* distortion and *reflection* effects. Each star heats the hemisphere of its companion that faces it, creating *hot spots* and brightness variations. The plotted light curve, however, shows nearly vertical ingress and egress and flat minima, which are inconsistent with real stellar atmospheres: *limb darkening* smooths the contacts and rounds the minima. The figure therefore depicts an idealized, "boxy" light curve rather than a realistic one.

d) A0 V stars are used as calibration "white standards", with $(B-V)_{\mathrm{A0}} = 0.00$ mag. For the G2 V component we adopt the solar value from the constants table, $(B-V)_{\mathrm{G2}} = 0.65$ mag.

The out-of-eclipse color index of the system is

$$(B-V)_{\mathrm{out}} = \cancelto{0}{(B-V)_{\mathrm{A0}}} + \Delta B - \Delta V.$$

Here ΔB and ΔV denote the changes in the system's magnitudes caused by the presence of the secondary component.

$$\Delta V = -2.5\lg\frac{F_{h,V}+F_{c,V}}{F_{h,V}} = 2.5\lg(1-\delta_{\rm sec}) = 2.5\lg(1-0.0255) = -0.028,$$

$$\begin{aligned}\Delta B &= -2.5\lg\frac{F_{h,B}+F_{c,B}}{F_{h,B}} = -2.5\lg\frac{F_{h,V}+F_{c,V}\cdot 10^{-0.4\times0.65}}{F_{h,V}} = \\ &= -2.5\lg\frac{I_{h,V}\,A_h + I_{c,V}\,A_c\cdot 10^{-0.4\times0.65}}{I_{h,V}\,A_h} = -2.5\lg\left(1+\frac{A_c\cdot 10^{-0.4\times0.65}}{k_V\,A_h}\right) = \\ &= -2.5\lg\left(1+\frac{0.207}{7.9}\times 10^{-0.4\times0.65}\right) = -0.0155.\end{aligned}$$

$$\therefore\ (B-V)_{\rm out} = 0.012\ \text{mag}.$$

At primary minimum the hotter star is partially eclipsed. The system becomes slightly redder:

$$\begin{aligned}\Delta V' &= -2.5\lg\frac{F_{h,V}\cdot\left(1-\frac{A_c}{A_h}\right)+F_{c,V}}{F_{h,V}} = -2.5\lg\frac{I_{h,V}\,A_h\cdot\left(1-\frac{A_c}{A_h}\right)+I_{c,V}\,A_c}{I_{h,V}\,A_h} = \\ &= -2.5\lg\left[\left(1-\frac{A_c}{A_h}\right)+\frac{A_c}{A_h}\cdot\frac{1}{k_V}\right] = -2.5\lg\left[1-\frac{A_c}{A_h}\left(1-\frac{1}{k_V}\right)\right] = \\ &= -2.5\lg\left[1-0.207\times\frac{6.9}{7.9}\right] = 0.217,\end{aligned}$$

$$\begin{aligned}\Delta B' &= -2.5\lg\frac{F_{h,B}\cdot\left(1-\frac{A_c}{A_h}\right)+F_{c,B}}{F_{h,B}} = -2.5\lg\frac{F_{h,V}\cdot\left(1-\frac{A_c}{A_h}\right)+F_{c,V}\cdot 10^{-0.4\times0.65}}{F_{h,V}} = \\ &= -2.5\lg\frac{I_{h,V}\,A_h\cdot\left(1-\frac{A_c}{A_h}\right)+I_{c,V}\,A_c\cdot 10^{-0.4\times0.65}}{I_{h,V}\,A_h} = \\ &= -2.5\lg\left[\left(1-\frac{A_c}{A_h}\right)+\frac{A_c}{A_h}\cdot\frac{10^{-0.4\times0.65}}{k_V}\right] = -2.5\lg\left[1-\frac{A_c}{A_h}\left(1-\frac{10^{-0.4\times0.65}}{k_V}\right)\right] = \\ &= -2.5\lg\left[1-0.207\times\left(1-\frac{10^{-0.4\times0.65}}{7.9}\right)\right] = 0.232.\end{aligned}$$

$$\therefore\ (B-V)_{\rm pri} = \Delta B' - \Delta V' = 0.015\ \text{mag}.$$

Brief consistency check: $\Delta V' = \Delta m_{\rm pri} - \Delta m_{\rm sec}$.

At secondary minimum only the A0 star remains visible, so $(B-V)_{\rm sec} = 0.00$ mag.

Hence the color index curve is nearly flat, with a tiny red "bump" at primary eclipse and a small blueward excursion (back to ≈ 0) at secondary eclipse (see Fig. 3).

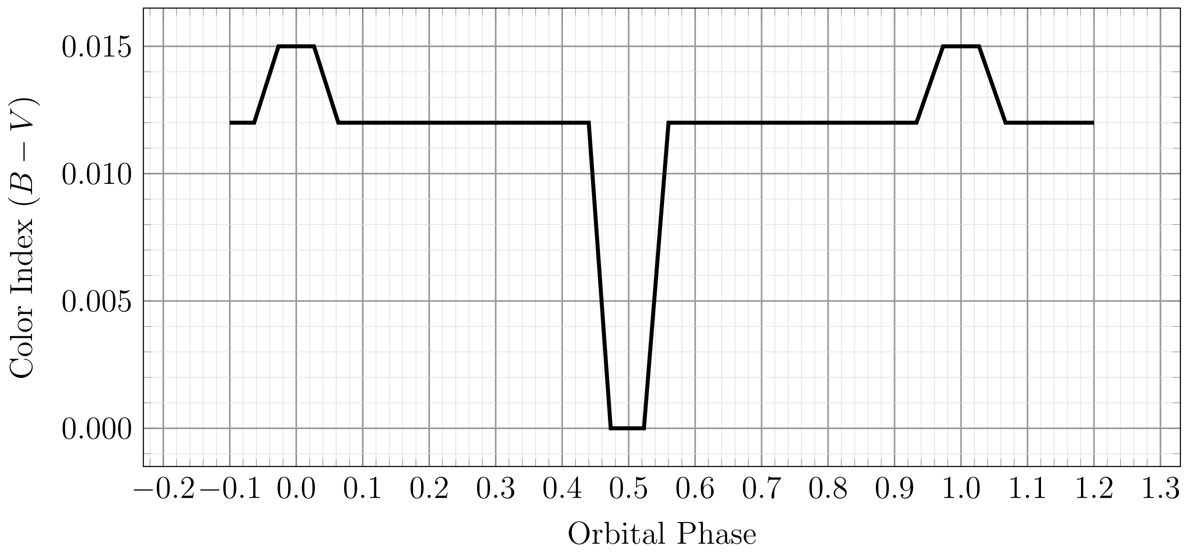


Figure 3: Sketch of the color index curve of the system

Let A0 V be the cooler star.

With $T_c = T_{\mathrm{A0}} = 9700$ K, the companion's temperature is $T_h = T_c \times 1.68 \approx 16.3 \times 10^3$ K, corresponding to a B3–B4 V star. Taking $R_{\mathrm{A0}} = 2.2\,R_\odot$, the radius ratio derived from the light curve minima gives

$$R_h = \frac{R_c}{0.455} \approx 4.8\,R_\odot.$$

However, a typical B3–B4 V star has $R_{\mathrm{B3\text{–}B4}} \approx 3.5\,R_\odot$, significantly smaller than the required value. Such stars have a significant bolometric correction, which strongly affects the temperature estimate. We adopt the bolometric correction approximation (see Fig. 4)

$$BC_V \approx -5.0\left(\frac{\lg T/\mathrm{K} - 4}{0.8}\right), \qquad T \geqslant 10^4\,\mathrm{K}.$$

The self-consistency condition relates the bolometric surface brightness ratio ($\propto T^4$) to the measured V band ratio k_V:

$$\frac{T_h^4}{T_{\mathrm{A0}}^4} = k_V \cdot 10^{-0.4\,BC_V(T_h)} = k_V\left(\frac{T_h}{10^4\,\mathrm{K}}\right)^{2.5}.$$

This yields a closed-form solution

$$T_h/\mathrm{K} = \left[\frac{(T_{\mathrm{A0}}/\mathrm{K})^4\,k_V}{10^{10}}\right]^{2/3} = \left[\frac{(9.7\cdot 10^3)^4 \times 7.9}{10^{10}}\right]^{2/3} \approx 3.7 \cdot 10^4.$$

This corresponds to a very hot star of spectral type O. Such stars have radii significantly larger than R_h (from about $7\,R_\odot$ upward). Therefore, even taking the bolometric correction into account does not allow further development of this case.

The only possible composition of the system is A0 V + G2 V.

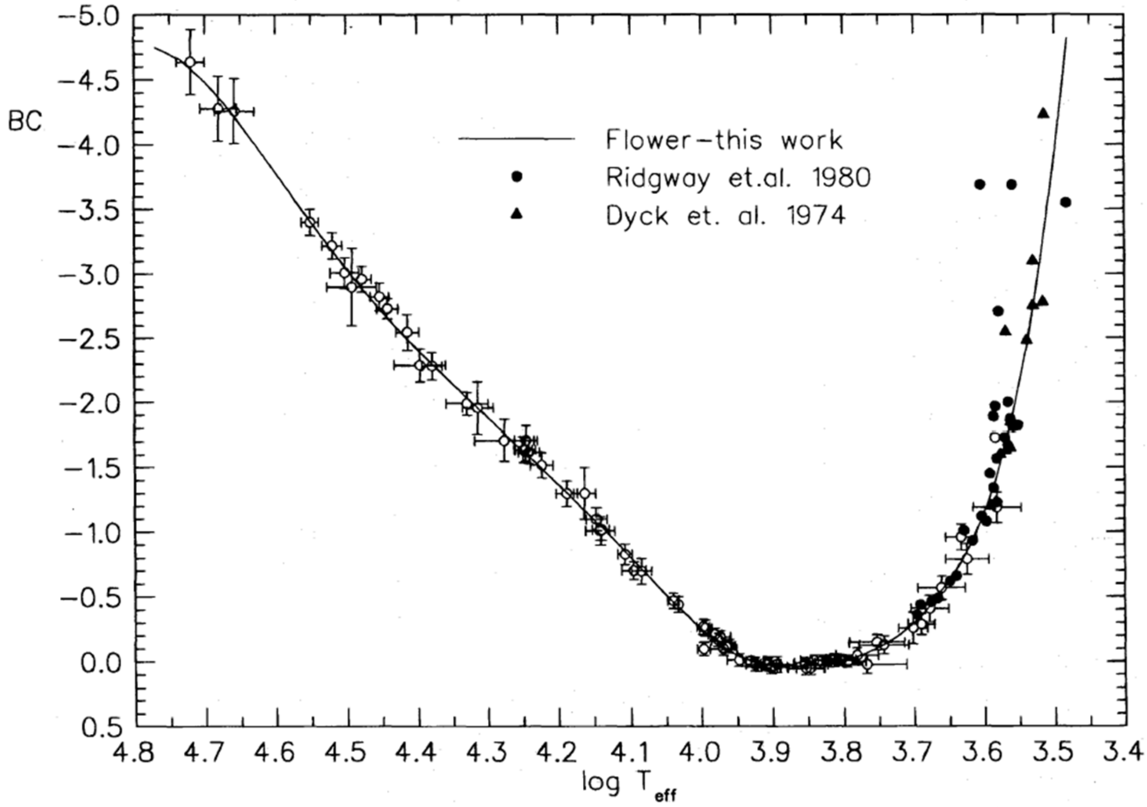


Figure 4: Bolometric correction – effective temperature relation according to Flower (1996)

Marking Scheme:

- Question (a) Spectral type of 2nd component
  1. Ratio of the stars' temperatures — **3 pt.**
  2. Spectral type — **2 pt.**
- Question (b) Distance between components
  1. Mass-luminosity relation or a comparable method — **2 pt.**
  2. Semi-major axis of the relative orbit (correct method + answer) — **2 pt.**
- Question (c) What's wrong
  1. Mention of stellar deformation — **1 pt.**
  2. Mention of hot spots — **1 pt.**
  3. Mention of limb darkening — **1 pt.**
- Question (d) Sketch for $(B-V)$
  1. Quantitative analysis of $(B-V)$ curve — **2 pt.**
  2. Schematic drawing — **3 pt.**
- Criterion (x) Consideration of the case where the A0 V star is the cooler component **3 pt.,** includes 1 pt. for mentioning this case

Constants

Universal

Gravitational constant	G	$= 6.67 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$
Speed of light	c	$= 3.00 \cdot 10^{8}$ m/s
Planck constant	h	$= 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	k_B	$= 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R}$	$= 8.314$ J/(mol · K)
Proton mass	m_p	$= 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au	$= 149.6 \cdot 10^{6}$ km
Parsec	1 pc	= 206 265 au
Hubble constant	H_0	= 70 (km/s)/Mpc

Earth

Radius	$R_\oplus$	= 6371 km
Mass	$\mathfrak{M}_\oplus$	$= 5.97 \cdot 10^{24}$ kg
Obliquity of ecliptic	ε	$= 23.4^\circ$
Surface gravity	g	$= 9.81$ m/s^2
Orbital period	$T_\oplus$	= 365.26 days
Orbital eccentricity	$e_\oplus$	= 0.0167

Hydrogen spectrum

Lyman Lα	$\lambda_{\mathrm{L}\alpha}$	= 1215.7 Å
Balmer Hα	$\lambda_{\mathrm{H}\alpha}$	= 6562.8 Å

Jupiter

Radius	R_J	$= 6.99 \cdot 10^{4}$ km
Mass	$\mathfrak{M}_J$	$= 1.90 \cdot 10^{27}$ kg
Orbital radius	a_J	= 5.20 au
Orbital period	T_J	= 11.86 yr

Sun

Radius	$R_\odot$	$= 6.96 \cdot 10^{5}$ km
Mass	$\mathfrak{M}_\odot$	$= 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot$	$= 4.74^{\mathrm{m}}$ (bol.)
Effective temperature	$T_\odot$	$= 5.8 \cdot 10^{3}$ K
Luminosity	$L_\odot$	$= 3.828 \cdot 10^{26}$ W
Color index	$(B-V)_\odot$	$= +0.65^{\mathrm{m}}$

Emission constants

Stefan–Boltzmann	σ	$= 5.67 \cdot 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}^4}$
Wien's displacement	b	= 2898 µm · K

UBV... system	Mean wavelengths	
U band	λ_U	= 364 nm
B band	λ_B	= 442 nm
V band	λ_V	= 540 nm
K band	λ_K	= 2190 nm