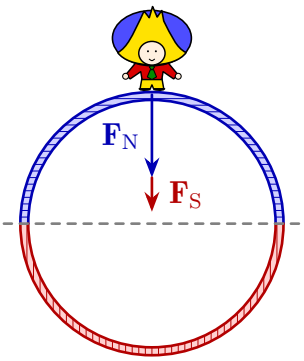
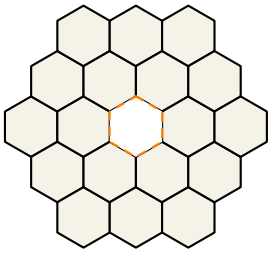
1 Dunno on a Hollow Moon



Dunno lands on a strange moon. It is a thin, uniform spherical shell, with no mass inside it. Dunno stands at the north pole of the moon.

The shell is divided into northern and southern hemispheres by its equator. Find the ratio of the magnitudes of the net gravitational forces $F_N : F_S$ exerted by each hemisphere separately.

2 Bee Mirror



A telescope's primary mirror is assembled from 18 regular hexagonal segments, each with side length 48 cm. The segments form a hexagonal mirror with a central hexagonal hole large enough to fit one more segment. Each segment has an active control system to compensate for thermal deformations.

a) Estimate the total reflecting area and the diameter of the mirror, defined as the distance between its two farthest-apart points. Treat the mirror as flat.

b) A star of apparent magnitude $m_V = +18$ is observed in the V band. The detector has quantum efficiency $\eta = 0.8$. Estimate how many photons it detects in a 10-minute exposure. Use 10^{10} photons $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ as the integrated V-band photon flux from a zero-magnitude star. Ignore atmospheric absorption and all losses except the detector's quantum efficiency.

c) The glass has a linear thermal expansion coefficient $\alpha = 5 \cdot 10^{-7}\ \mathrm{K}^{-1}$. At night, the glass cools by $\Delta T = 10$ K. Assume that the mirror contracts uniformly while retaining its shape, so its focal length scales with its linear dimensions. If the focal ratio is $f/D = 2.25$, where D is the mirror diameter from part (a), by how far and in which direction should the detector in the focal plane be moved to restore focus?

3 Solar Sail, Solar Snail

A spacecraft is in a circular orbit of radius r_1 around the Sun. Consider two ways of reaching an orbit with larger radius r_2:

a) The spacecraft performs a Hohmann transfer: one engine burn puts it on an elliptical orbit tangent to both circular orbits.

b) The spacecraft unfolds a perfectly reflecting solar sail without changing its velocity. The sail is kept perpendicular to the Sun's rays. The sail area is chosen so that the new orbit has apoapsis r_2.

Show that the sail-driven trip from r_1 to r_2 takes longer than the Hohmann transfer.

4 git merge --squash

On January 14, 2025, the LIGO gravitational-wave observatories detected GW250114, a signal produced by the merger of two black holes, which had masses of approximately $m_1 = 33.6\,\mathfrak{M}_\odot$ and $m_2 = 32.2\,\mathfrak{M}_\odot$. They merged to form a single rotating black hole.

This event provided strong observational evidence supporting Hawking's area theorem (1971), which states that the total area of black hole event horizons cannot decrease.

The event horizon area of a rotating black hole is given by

$$\mathcal{A} = \frac{8\pi G^2 m^2}{c^4}\left(1 + \sqrt{1-\chi^2}\right),$$

where m is the black hole mass and $\chi \in [0; 1]$ is its dimensionless spin parameter. The spin parameter ranges from 0 for a non-rotating black hole to 1 for a maximum rotating black hole.

a) Show that energy conservation imposes an upper limit on the fractional increase in the total event horizon area:

$$\frac{\mathcal{A}_f - \mathcal{A}_i}{\mathcal{A}_i} \leqslant 3.$$

Under what conditions could this upper limit be approached?

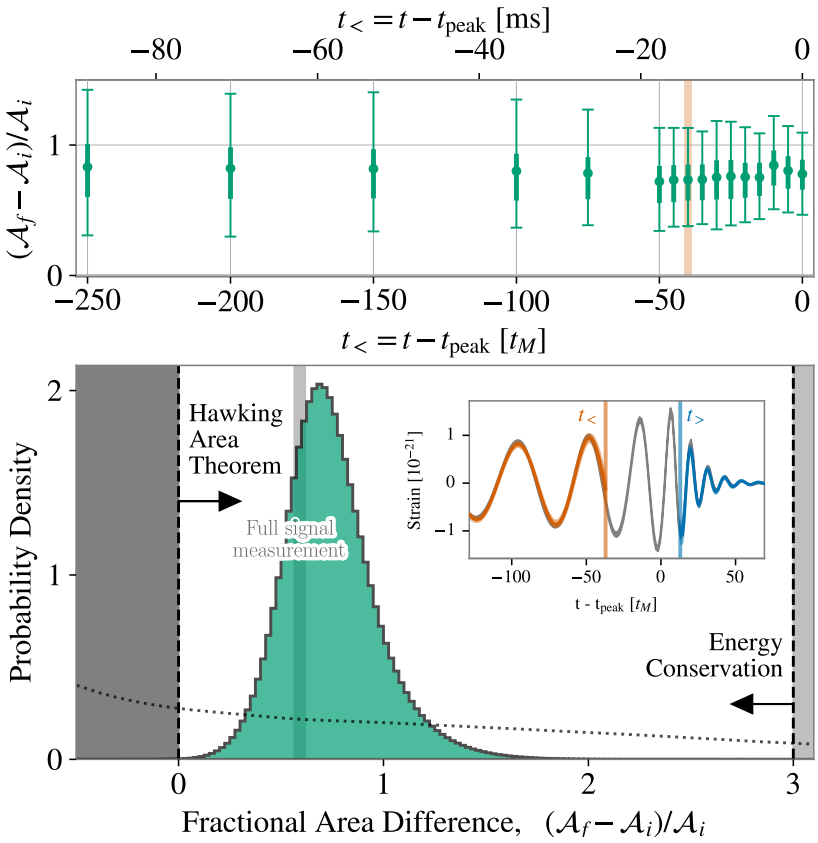


Figure 1: Fractional difference between the area of the final black hole and the initial black holes. Credit: A. G. Abac et al. (LIGO Scientific, Virgo, and KAGRA Collaborations). DOI: `https://doi.org/10.1103/kw5g-d732`

Fig. 1 presents the results of the analysis of GW250114. In the lower panel, the horizontal axis represents the fractional change in the total event horizon area,

$$\frac{\Delta\mathcal{A}}{\mathcal{A}_i} = \frac{\mathcal{A}_f - \mathcal{A}_i}{\mathcal{A}_i},$$

where $\mathcal{A}_i = \mathcal{A}_1 + \mathcal{A}_2$ is the sum of the initial horizon areas and $\mathcal{A}_f$ is the horizon area of the final black hole.

The green histogram shows the probability distribution obtained by analysing the gravitational-wave signal before and after the merger separately. The narrow vertical gray band indicates the 90 % credible interval obtained from an analysis of the entire signal.

b) The initial black holes had small spins, which can be neglected in the calculations below. The final black hole has a dimensionless spin of approximately $\chi_f = 0.68$.

   Using the gray band in the lower panel of the figure, estimate the mass of the final black hole. How much energy was emitted in gravitational waves, expressed in units of $\mathfrak{M}_\odot c^2$?

c) Roughly estimate the frequency of the gravitational waves shortly before the merger. Hence, determine the orbital frequency of the two black holes, given that the dominant gravitational-wave frequency is twice the orbital frequency.

   The time axis is expressed in units of

$$t_M = \frac{G(m_1 + m_2)(1+z)}{c^3},$$

   where $z = 0.09$ is the cosmological redshift of the source.

d) The dimensionless amplitude of the gravitational waves reaching Earth can be estimated as

$$h \simeq \frac{4G^2 m_1 m_2}{c^4 D r}.$$

   Here r is the separation between the black holes, D is the distance to the system.

   Estimate the gravitational-wave amplitude h at Earth and the signal-to-noise ratio (SNR) of the detection, assuming an effective detector noise amplitude of $h_{\text{noise}} = 1.4 \cdot 10^{-23}$.

5 A Spectre Is Haunting Europa

In 1848, Karl Marx and Friedrich Engels wrote, *"A spectre is haunting Europe — the spectre of Communism."*

Suppose a translucent spectre (= ghost) wanders over the surface of Jupiter's moon Europa and sometimes appears on the side facing Earth. The ghost reflects half the light falling on it and transmits the other half. Assume diffuse reflection and transmission.

The largest telescope available at the time was the Leviathan of Parsonstown, completed in 1845, with an aperture of 1.83 m. Could an astronomer detect the spectre of communism on Europa?

6 Interstellar Cloud Computing

Astronomers observe two planetary nebulae. Model each nebula as a uniform, optically thin sphere of ionized gas, and assume both nebulae have the same temperature and composition. The volume emissivity (power emitted per unit volume) in the Hβ line is proportional to the square of the electron number density: $\varepsilon_{\mathrm{H}\beta} \propto n_e^2$.

The measured properties are shown below:

		Nebula A	**Nebula B**
Angular diameter	θ ($''$)	40	20
Hβ line flux	F (10^{-10} erg cm^{-2} s^{-1})	1.0	1.3
Ionized gas mass	$\mathfrak{M}$ ($\mathfrak{M}_\odot$)	0.28	0.10

An independent method gives the distance to Nebula A as $d_A = 1.0$ kpc. Estimate the distance to Nebula B. Ignore interstellar extinction.

7 Ulysses' Quick Visit

The Ulysses spacecraft was launched toward Jupiter in October 1990. It left Earth tangentially to Earth's orbit and then moved under the Sun's gravity until it reached Jupiter. After a passive gravity-assist maneuver, it entered a polar orbit around the Sun with eccentricity $e = 0.6$. Assume that Ulysses was at the aphelion of this orbit after its flyby of Jupiter and was heading toward the Sun's south pole.

What was the latitude of the point on Jupiter's surface that came closest to the spacecraft during the flyby? Neglect the tilt of Jupiter's rotation axis.

8 Times Square

The table gives the present-day heliocentric positions and velocity components of two stars:

$\boldsymbol{\alpha}$	$\boldsymbol{\delta}$	$\boldsymbol{r}$ **(pc)**	$\boldsymbol{\mu_\alpha}$ **(mas yr^{-1})**	$\boldsymbol{\mu_\delta}$ **(mas yr^{-1})**	$\boldsymbol{v_r}$ **(km s^{-1})**
$14^{\mathrm{h}}\,00^{\mathrm{m}}\,00.000^{\mathrm{s}}$	$0^\circ\,00'\,00''$	3.250	-90	0	0
$09^{\mathrm{h}}\,00^{\mathrm{m}}\,00.000^{\mathrm{s}}$	$20^\circ\,00'\,00''$	4.750	0	-180	0

Find how close the stars will come to each other and how many years from now that will happen. Assume each star's heliocentric velocity is constant in space.

9 The Hitchhiker's Guide to the Galactic Center

Arthur Dent and Ford Prefect are looking at a radio image of the Milky Way's center. Draw the Galactic equator on the image *as accurately as possible.* Mark the direction toward the North Galactic Pole with an arrow. Justify your drawing.

Hint. The source marked Tornado (SNR) has galactic latitude $b \simeq -0.1^\circ$.

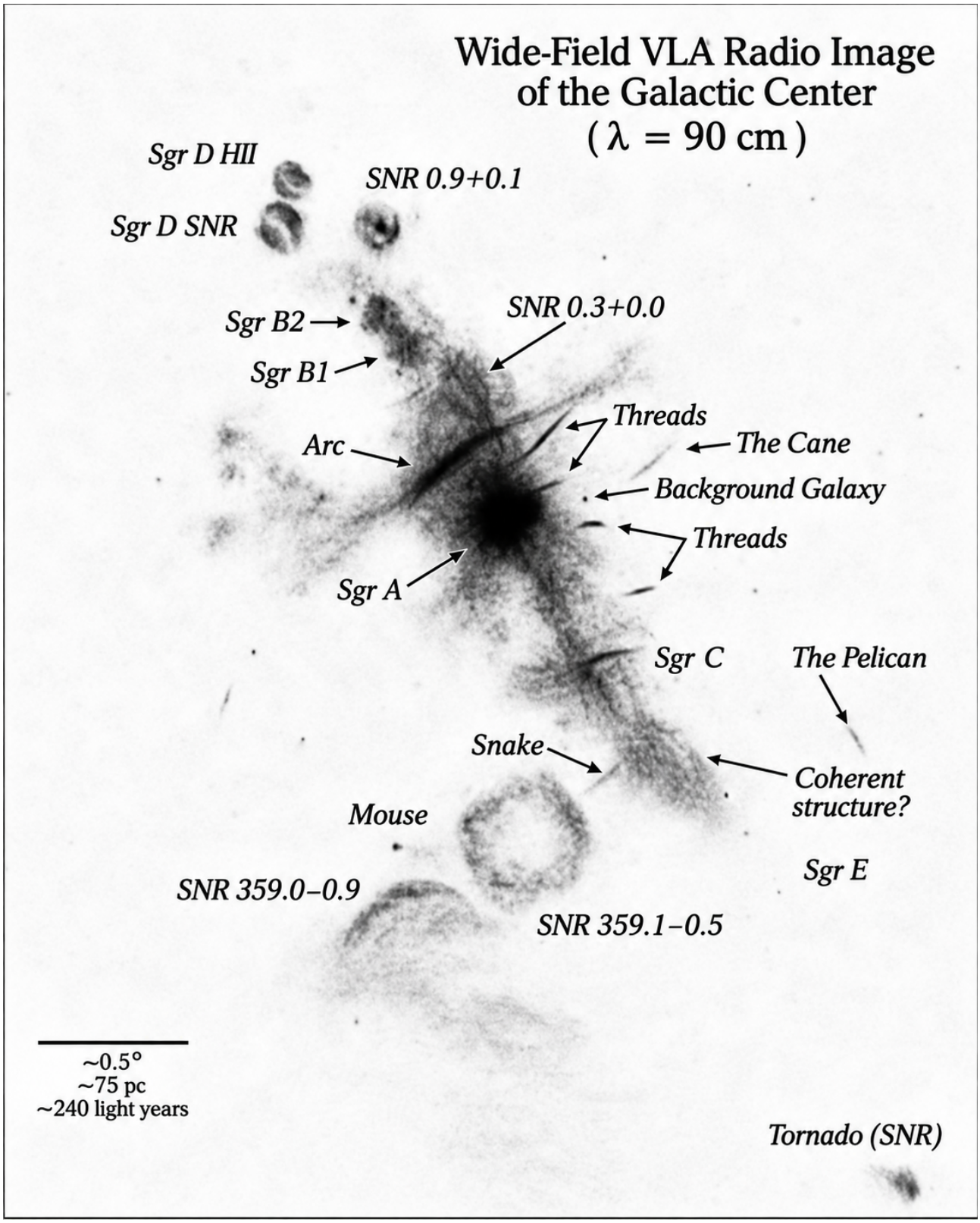


Figure 2: A 90 cm radio image of the Galactic center region by the Very Large Array. Credit: NRAO/AUI/NSF and N. E. Kassim, Naval Research Laboratory. Black-and-white version for printing and drawing

10 You Spin Me Round

Radio telescopes observe an isolated pulsar, a rapidly rotating neutron star with a spin period of $P = 0.1$ s. After several years of observations, its period derivative is measured to be $\dot{P} = 10^{-15}$ (in units of s/s), meaning its period increases by 10^{-15} seconds each second.

The neutron star has mass $\mathfrak{M} \approx 1.4\mathfrak{M}_\odot$ and radius $R \approx 10$ km. Model the star as a uniform sphere.

The pulsar loses rotational energy through magnetic-dipole radiation. The power radiated by a rotating magnetic dipole with magnetic moment μ and angular velocity Ω is

$$\dot{E}_{\mathrm{md}} = \frac{2}{3c^3}\mu^2\Omega^4 \sin^2\alpha,$$

where α is the angle between the magnetic and rotation axes, and c is the speed of light. Assume that the magnetic moment, the angle α, and the moment of inertia remain constant.

a) Show that magnetic-dipole braking implies $P\dot{P} = \text{const}$.

b) Estimate the neutron star's moment of inertia I.

c) Find the positive rate of rotational energy loss, $\dot{E}_{\mathrm{rot}} \equiv -dE_{\mathrm{rot}}/dt$.

d) Estimate the pulsar's age τ, assuming it was born with a period $P_0 \ll P$.

Constants

Universal

Gravitational constant	$G = 6.67 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$
Speed of light	$c = 3.00 \cdot 10^8$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R} = 8.314$ J/(mol · K)
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au $= 149.6 \cdot 10^6$ km
Parsec	1 pc $= 206\,265$ au
Hubble constant	$H_0 = 70$ (km/s)/Mpc

Earth

Radius	$R_\oplus = 6371$ km
Mass	$\mathfrak{M}_\oplus = 5.97 \cdot 10^{24}$ kg
Obliquity of ecliptic	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26$ days
Orbital eccentricity	$e_\oplus = 0.0167$

Jupiter

Radius	$R_J = 6.99 \cdot 10^4$ km
Mass	$\mathfrak{M}_J = 1.90 \cdot 10^{27}$ kg
Orbital radius	$a_J = 5.20$ au
Orbital period	$T_J = 11.86$ yr

Europa

Radius	$R_E = 1.56 \cdot 10^3$ km
Albedo	$A_E = 0.7$ (approx.)

Sun

Radius	$R_\odot = 6.96 \cdot 10^5$ km
Mass	$\mathfrak{M}_\odot = 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot = 4.74^{\mathrm{m}}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^3$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Emission constants

Stefan–Boltzmann	$\sigma = 5.67 \cdot 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}^4}$
Wien's displacement	$b = 2898\ \mu\mathrm{m} \cdot \mathrm{K}$