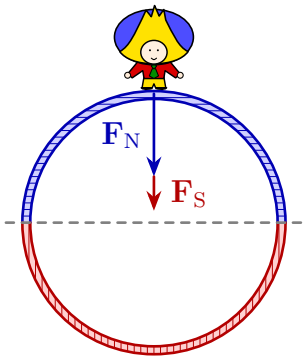
1 Dunno on a Hollow Moon



Dunno lands on a strange moon. It is a thin, uniform spherical shell, with no mass inside it. Dunno stands at the north pole of the moon.

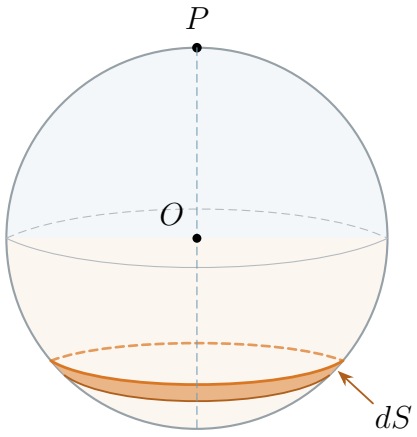
The shell is divided into northern and southern hemispheres by its equator. Find the ratio of the magnitudes of the net gravitational forces $F_N : F_S$ exerted by each hemisphere separately.

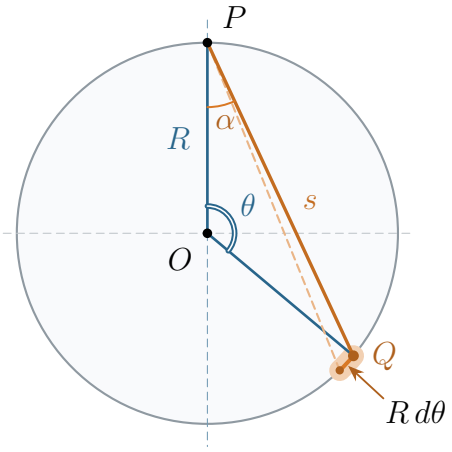
Solution:

Let the shell have radius R and mass $\mathfrak{M}$, and let Dunno's mass be m. We first find the force from the southern hemisphere. A ring at polar angle θ has area

$$dS = 2\pi R \sin\theta \cdot R\, d\theta = 2\pi R^2 \sin\theta\, d\theta.$$

The mass per unit area is constant, $\sigma = \mathfrak{M}/(4\pi R^2)$.





Let s be the distance from Dunno at the north pole to the ring. Geometry gives

$$s^2 = 2R^2(1-\cos\theta),$$
$$s\, ds = R^2 \sin\theta\, d\theta.$$

The ring mass can now be written in terms of s:

$$d\mathfrak{M} = \sigma\, dS = 2\pi R^2 \sigma \sin\theta\, d\theta = 2\pi\sigma s\, ds.$$

The angle between the force toward the ring's pieces and the direction toward the moon's centre has cosine $R(1-\cos\theta)/s = s/(2R)$. Hence the component of its force along the axis PO is

$$dF = \frac{Gm\, d\mathfrak{M}}{s^2}\frac{s}{2R} = \frac{\pi G m \sigma}{R}\, ds.$$

For the southern hemisphere $\sqrt{2}R \leqslant s \leqslant 2R$, giving

$$F_S = \frac{\pi G m \sigma}{R} \int_{\sqrt{2}R}^{2R} ds = \pi G m \sigma \left(2 - \sqrt{2}\right). \tag{1}$$

For Dunno standing just outside the shell, the shell theorem gives the force of the *whole* shell as

$$F_{\text{all}} = \frac{G\mathfrak{M}m}{R^2} = 4\pi G m \sigma.$$

Subtracting the distant hemisphere's force gives

$$F_N = F_{\text{all}} - F_S = \pi G m \sigma \left(2 + \sqrt{2}\right).$$

Both forces point toward the moon's centre, so their magnitudes have ratio

$$\frac{F_N}{F_S} = \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = 3 + 2\sqrt{2}.$$

Alternative approach. Take the z axis along the symmetry axis, with the shell centre at $z = 0$ and the north pole at $z = R$. To keep the surface limit well defined, place Dunno at height $h > 0$ above the pole and write $H = R + h$ for his distance from the centre.

A horizontal ring at coordinate z has radius $\rho = \sqrt{R^2 - z^2}$. Its width along the spherical surface is

$$d\ell = \sqrt{1 + \left(\frac{d\rho}{dz}\right)^2}\, dz = \frac{R}{\rho}\, dz,$$

$$dS = 2\pi\rho\, d\ell = 2\pi R\, dz.$$

Thus $d\mathfrak{M} = 2\pi\sigma R\, dz$. Every point of the ring is at the same distance from Dunno,

$$s(z) = \sqrt{\rho^2 + (H - z)^2} = \sqrt{R^2 + H^2 - 2Hz}.$$

The transverse force components cancel. The projection onto the downward axis introduces the factor $(H - z)/s$, so

$$dF = 2\pi G m \sigma R \frac{H - z}{(R^2 + H^2 - 2Hz)^{3/2}}\, dz.$$

The southern hemisphere corresponds to $-R \leqslant z \leqslant 0$, and the northern one to $0 \leqslant z \leqslant R$. With $u = s(z)$ and $dz = -u\, du/H$, the force from an arbitrary interval is

$$F(z_1, z_2) = \frac{\pi G m \sigma R}{H^2} \left[u - \frac{H^2 - R^2}{u}\right]_{s(z_2)}^{s(z_1)}.$$

Using $s(-R) = H + R$, $s(0) = \sqrt{H^2 + R^2}$, and $s(R) = H - R$ gives

$$F_S(H) = \frac{2\pi G m \sigma R^2}{H^2}\left(1 - \frac{R}{\sqrt{H^2+R^2}}\right),$$
$$F_N(H) = \frac{2\pi G m \sigma R^2}{H^2}\left(1 + \frac{R}{\sqrt{H^2+R^2}}\right).$$

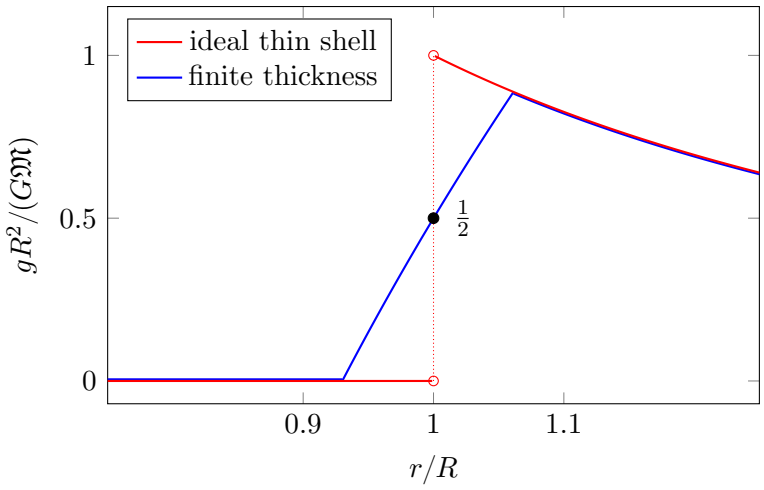
Their sum is

$$F_S(H) + F_N(H) = \frac{4\pi G m \sigma R^2}{H^2} = \frac{G\mathfrak{M}m}{H^2},$$

as required by the shell theorem. Taking the limit $H \to R + 0$ now yields

$$F_S \to \pi G m \sigma \left(2 - \sqrt{2}\right),$$
$$F_N \to \pi G m \sigma \left(2 + \sqrt{2}\right),$$
$$\frac{F_N}{F_S} = 3 + 2\sqrt{2}.$$

Note on the surface limit. The same ring integral (1) cannot be applied directly to the northern hemisphere with Dunno exactly at $s = 0$. It would give $F_N^{\text{direct}} = \pi G m \sigma \sqrt{2}$ and thus only $F_N^{\text{direct}} + F_S = 2\pi G m \sigma = F_{\text{all}}/2$ for the entire shell. The ideal zero-thickness shell has *a discontinuity:* its gravitational field is zero just inside and $G\mathfrak{M}/R^2$ just outside. The direct surface integral gives the midpoint of these two limits:



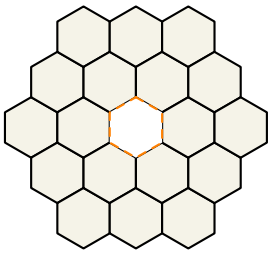
$$g(r) = \frac{G\mathfrak{M}}{r^2}\begin{cases} 0, & r < a, \\ \dfrac{r^3 - a^3}{b^3 - a^3}, & a \leqslant r \leqslant b, \\ 1, & r > b. \end{cases}$$

As the thickness tends to zero, the blue transition approaches the red jump. Dunno stands outside, so the relevant limit is $h \to 0^+$; the tiny patch immediately below him adds the missing $2\pi G m \sigma$.

Marking Scheme (20 points):

A	Use of axial symmetry and surface mass density, or an equivalent coefficient	**2**
B	Division of the shell into thin rings and a correct expression for the ring area	**3**
C	Expression for the distance from Dunno to a ring element	**2**
D	Correct projection of the gravitational force onto the symmetry axis	**3**
E	Correct integration limits for the southern hemisphere and calculation of F_S	**3**
F	Total gravitational attraction of the hollow moon, $F_{\text{all}} = G\mathfrak{M}m/R^2$	**3**
G	Expression for F_N or an equivalent direct calculation	**2**
H	Derivation and simplification of the force ratio, $F_N/F_S = 3 + 2\sqrt{2} \simeq 5.83$	**2**

2 Bee Mirror



A telescope's primary mirror is assembled from 18 regular hexagonal segments, each with side length 48 cm. The segments form a hexagonal mirror with a central hexagonal hole large enough to fit one more segment. Each segment has an active control system to compensate for thermal deformations.

a) Estimate the total reflecting area and the diameter of the mirror, defined as the distance between its two farthest-apart points. Treat the mirror as flat.

b) A star of apparent magnitude $m_V = +18$ is observed in the V band. The detector has quantum efficiency $\eta = 0.8$. Estimate how many photons it detects in a 10-minute exposure. Use 10^{10} photons $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ as the integrated V-band photon flux from a zero-magnitude star. Ignore atmospheric absorption and all losses except the detector's quantum efficiency.

c) The glass has a linear thermal expansion coefficient $\alpha = 5 \cdot 10^{-7}\ \mathrm{K}^{-1}$. At night, the glass cools by $\Delta T = 10$ K. Assume that the mirror contracts uniformly while retaining its shape, so its focal length scales with its linear dimensions. If the focal ratio is $f/D = 2.25$, where D is the mirror diameter from part (a), by how far and in which direction should the detector in the focal plane be moved to restore focus?

Solution:

a) Let $L = 0.48$ m be the side length of a segment. Each regular hexagon consists of six equilateral triangles, so its area is

$$\frac{\sqrt{3}L^2}{4} \times 6 = \frac{3\sqrt{3}}{2}L^2.$$

The central cell is absent; there are 18 reflecting cells. Thus

$$A = \frac{3\sqrt{3}}{2}L^2 \times 18 = 27\sqrt{3}L^2 \simeq 10.8\ \mathrm{m}^2.$$

In the diagram, $AB = 5\sqrt{3}L$, $BC = L$, and $AB \perp BC$. Opposite pairs of outer vertices of this type are farthest apart. Therefore the diameter defined in the question is

$$D = AC = \sqrt{(5\sqrt{3}L)^2 + L^2} = \sqrt{76}\,L \simeq 4.18\ \mathrm{m}.$$

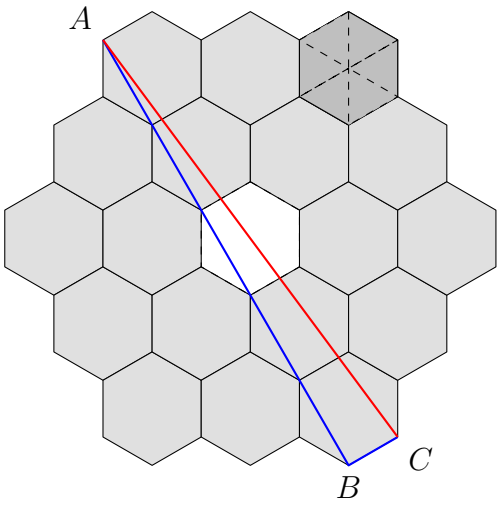


Figure 1: One of the longest diagonals AC of the mirror

b) The given zero-magnitude photon flux supplies the required V-band calibration directly. For $m_V = 18$ it becomes

$$\Phi_V = 10^{10} \times 10^{-0.4 m_V} = 10^{10} \times 10^{-7.2} \simeq 6.31 \cdot 10^2 \text{ photons m}^{-2}\,\text{s}^{-1}.$$

During $t = 600$ s, the number of photons detected is

$$N = \eta A t \Phi_V \simeq 0.8 \times 10.8 \text{ m}^2 \times 600 \text{ s} \times 6.31 \cdot 10^2 \text{ m}^{-2}\,\text{s}^{-1} \simeq 3.3 \cdot 10^6.$$

c) Initially $f = 2.25D \simeq 9.42$ m. A temperature drop of 10 K reduces every linear dimension by the fraction $\alpha \Delta T = 5 \cdot 10^{-6}$. Hence

$$f' = f(1 - \alpha \Delta T),$$
$$f - f' = f \alpha \Delta T \simeq 4.7 \cdot 10^{-5} \text{ m} = 47 \text{ μm}.$$

The detector must move *toward the mirror,* because the new focal length is shorter.

Marking Scheme (20 points):

- **Question (a1) Reflecting area**
  - Calculation of the reflecting area as the area of one hexagonal segment multiplied by 18 ..... **2**
  - Correct reflecting area, approximately 10.8 m^2 ..... **2**
  - **A1 Subtotal 4**
- **Question (a2) Mirror diameter**
  - A reasonable geometric method for estimating the largest diameter **2**
  - Correct diameter, approximately 4 m (the exact geometry gives 4.18 m) **2**
  - **A2 Subtotal 4**
- **Question (b) Detected photons**
  - Use of the photon-flux relation $\Phi = \Phi_0\, 10^{-0.4m}$ **2**
  - Allowance for the reflecting area **1**
  - Allowance for the exposure time **1**
  - Allowance for the quantum efficiency $\eta = 0.8$ **2**
  - **B Subtotal 6**
- **Question (c) Refocusing after cooling**
  - Use of the focal ratio $f = 2.25D$ **2**
  - Use of the focal-length change $|\Delta f| = f \alpha\, |\Delta T|$ **2**
  - Correct direction of detector motion: toward the mirror **2**
  - **C Subtotal 6**

3 Solar Sail, Solar Snail

A spacecraft is in a circular orbit of radius r_1 around the Sun. Consider two ways of reaching an orbit with larger radius r_2:

a) The spacecraft performs a Hohmann transfer: one engine burn puts it on an elliptical orbit tangent to both circular orbits.

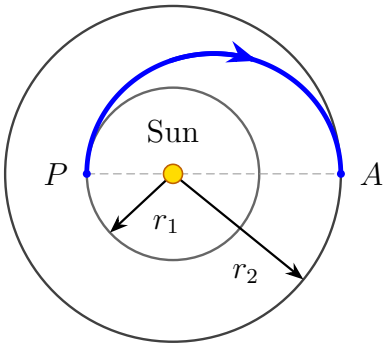
b) The spacecraft unfolds a perfectly reflecting solar sail without changing its velocity. The sail is kept perpendicular to the Sun's rays. The sail area is chosen so that the new orbit has apoapsis r_2.

Show that the sail-driven trip from r_1 to r_2 takes longer than the Hohmann transfer.

Solution:

a) Let $\mu = G\mathfrak{M}_\odot$ and $a = (r_1 + r_2)/2$. The Hohmann transfer follows half of an ellipse with semimajor axis a. Its travel time is

$$t_\mathrm{H} = \pi\sqrt{\frac{a^3}{\mu}}.$$



b) The sail faces the Sun throughout the trip. Solar radiation pressure therefore produces an outward force proportional to $1/r^2$. The net central force has the same form as gravity, but with a smaller parameter $\mu_\mathrm{s} = G\mathfrak{M}_\mathrm{eff} < \mu$. Immediately after deployment the spacecraft still has its original circular speed $v_1 = \sqrt{\mu/r_1}$. This is the speed at perihelion of the new ellipse. The vis-viva equation at r_1 gives

$$\frac{\mu}{r_1} = \mu_\mathrm{s}\left(\frac{2}{r_1} - \frac{1}{a}\right) = \frac{2\mu_\mathrm{s} r_2}{r_1(r_1 + r_2)},$$

$$\mu_\mathrm{s} = \mu\frac{r_1 + r_2}{2r_2}.$$

Since both transfers cover half an ellipse with the same semimajor axis,

$$t_\mathrm{s} = \pi\sqrt{\frac{a^3}{\mu_\mathrm{s}}},$$

$$\frac{t_\mathrm{s}}{t_\mathrm{H}} = \sqrt{\frac{\mu}{\mu_\mathrm{s}}} = \sqrt{\frac{2r_2}{r_1 + r_2}} > 1 \quad (r_2 > r_1).$$

Thus the sail reaches r_2 later than the Hohmann spacecraft. The inequality is strict for every outward transfer and becomes an equality only when $r_2 = r_1$.

Marking Scheme (20 points):

- **Question (a) Hohmann transfer**
  - **A1** Correct Hohmann transfer geometry **2**
  - **A2** Derivation of the transfer time, $t_{\mathrm{H}} = \pi\sqrt{a^3/\mu}$ **4**
- **Question (b) Solar sail and comparison of travel times**
  - **B1** Introduction of an effective solar mass or gravitational parameter for radiation pressure **4**
  - **B2** Expression for the effective parameter in terms of r_1 and r_2 ..... **4**
  - **B3** Correct sail travel time, $t_{\mathrm{s}} = \pi\sqrt{a^3/\mu_{\mathrm{s}}}$ **4**
  - **B4** Correct comparison of the travel times and conclusion that the Hohmann transfer is faster for $r_2 > r_1$ **2**

4 git merge --squash

On January 14, 2025, the LIGO gravitational-wave observatories detected GW250114, a signal produced by the merger of two black holes, which had masses of approximately $m_1 = 33.6\,\mathfrak{M}_\odot$ and $m_2 = 32.2\,\mathfrak{M}_\odot$. They merged to form a single rotating black hole.

This event provided strong observational evidence supporting Hawking's area theorem (1971), which states that the total area of black hole event horizons cannot decrease.

The event horizon area of a rotating black hole is given by

$$\mathcal{A} = \frac{8\pi G^2 m^2}{c^4}\left(1 + \sqrt{1-\chi^2}\right),$$

where m is the black hole mass and $\chi \in [0;\,1]$ is its dimensionless spin parameter. The spin parameter ranges from 0 for a non-rotating black hole to 1 for a maximum rotating black hole.

a) Show that energy conservation imposes an upper limit on the fractional increase in the total event horizon area:

$$\frac{\mathcal{A}_f - \mathcal{A}_i}{\mathcal{A}_i} \leqslant 3.$$

Under what conditions could this upper limit be approached?

Fig. 2 presents the results of the analysis of GW250114. In the lower panel, the horizontal axis represents the fractional change in the total event horizon area,

$$\frac{\Delta\mathcal{A}}{\mathcal{A}_i} = \frac{\mathcal{A}_f - \mathcal{A}_i}{\mathcal{A}_i},$$

where $\mathcal{A}_i = \mathcal{A}_1 + \mathcal{A}_2$ is the sum of the initial horizon areas and $\mathcal{A}_f$ is the horizon area of the final black hole.

The green histogram shows the probability distribution obtained by analysing the gravitational-wave signal before and after the merger separately. The narrow vertical gray band indicates the 90 % credible interval obtained from an analysis of the entire signal.

b) The initial black holes had small spins, which can be neglected in the calculations below. The final black hole has a dimensionless spin of approximately $\chi_f = 0.68$.

Using the gray band in the lower panel of the figure, estimate the mass of the final black hole. How much energy was emitted in gravitational waves, expressed in units of $\mathfrak{M}_\odot c^2$?

c) Roughly estimate the frequency of the gravitational waves shortly before the merger. Hence, determine the orbital frequency of the two black holes, given that the dominant gravitational-wave frequency is twice the orbital frequency.

The time axis is expressed in units of

$$t_M = \frac{G(m_1 + m_2)(1+z)}{c^3},$$

where $z = 0.09$ is the cosmological redshift of the source.

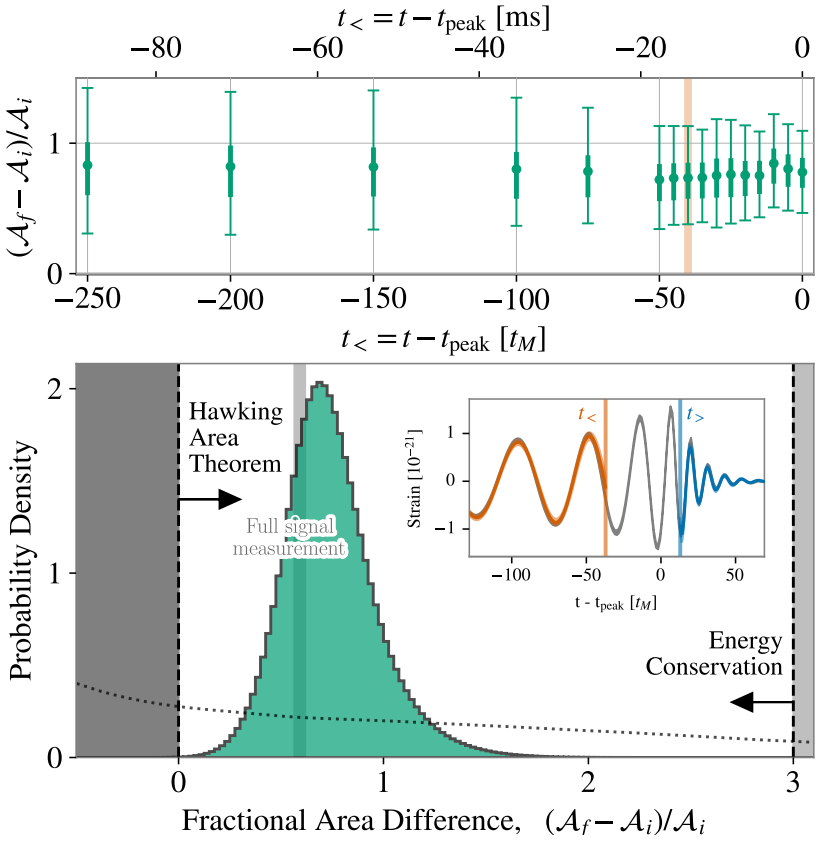


Figure 2: Fractional difference between the area of the final black hole and the initial black holes. Credit: A. G. Abac et al. (LIGO Scientific, Virgo, and KAGRA Collaborations). DOI: `https://doi.org/10.1103/kw5g-d732`

d) The dimensionless amplitude of the gravitational waves reaching Earth can be estimated as

$$h \simeq \frac{4G^2 m_1 m_2}{c^4 D r}.$$

Here r is the separation between the black holes, D is the distance to the system.

Estimate the gravitational-wave amplitude h at Earth and the signal-to-noise ratio (SNR) of the detection, assuming an effective detector noise amplitude of $h_{\text{noise}} = 1.4 \cdot 10^{-23}$.

Solution:

a) For a black hole of mass m, the given area formula implies

$$\frac{8\pi G^2 m^2}{c^4} \leqslant \mathcal{A} \leqslant \frac{16\pi G^2 m^2}{c^4}.$$

The initial area $\mathcal{A}_i = \mathcal{A}_1 + \mathcal{A}_2$ is at least $8\pi G^2(m_1^2 + m_2^2)/c^4$. The final mass cannot exceed $m_1 + m_2$ if no energy enters the system, and the final area cannot exceed that of a non-rotating black hole of that mass.

Since $(m_1 + m_2)^2 \leqslant 2(m_1^2 + m_2^2)$,

$$\frac{\mathcal{A}_f}{\mathcal{A}_i} \leqslant \frac{16\pi G^2(m_1 + m_2)^2/c^4}{8\pi G^2(m_1^2 + m_2^2)/c^4} \leqslant 4, \qquad \frac{\mathcal{A}_f - \mathcal{A}_i}{\mathcal{A}_i} \leqslant 3.$$

Approaching the limit would require nearly equal initial masses, nearly maximal initial spins, a nearly non-rotating final black hole, and negligible energy radiated away. These requirements describe a limiting case, not GW250114.

b) The centre of the narrow gray band in the lower panel is at about $x = \Delta\mathcal{A}/\mathcal{A}_i \simeq 0.58$. Neglecting the initial spins, $\mathcal{A}_i = 16\pi G^2(m_1^2 + m_2^2)/c^4$. Therefore

$$1 + x = \frac{\mathcal{A}_f}{\mathcal{A}_i} = \frac{m_f^2\left(1 + \sqrt{1 - \chi_f^2}\right)}{2(m_1^2 + m_2^2)},$$

$$m_f = \sqrt{\frac{2(1 + x)(m_1^2 + m_2^2)}{1 + \sqrt{1 - \chi_f^2}}}.$$

With $\chi_f = 0.68$, $m_1 = 33.6\,\mathfrak{M}_\odot$, and $m_2 = 32.2\,\mathfrak{M}_\odot$, this gives

$$m_f \simeq 63\,\mathfrak{M}_\odot,$$
$$E_{\mathrm{GW}} = (m_1 + m_2 - m_f)c^2 \simeq 3\,\mathfrak{M}_\odot c^2.$$

The graph supports approximate values rather than a precise final digit.

c) The time unit on the waveform inset is

$$t_M = \frac{G(m_1 + m_2)(1 + z)}{c^3} \simeq 3.5 \cdot 10^{-4}\ \mathrm{s}.$$

Consecutive crests near the end of the inspiral are separated by roughly $30t_M$, or about 0.01 s. Thus the gravitational-wave frequency is of order $f_{\mathrm{GW}} \sim 10^2$ Hz. The orbital frequency is half as large:

$$f_{\mathrm{orb}} \sim 50\ \mathrm{Hz}.$$

The frequency rises rapidly near the merger, so these are rough values.

d) Estimate the source distance from the small-redshift Hubble law:

$$D \simeq \frac{cz}{H_0} \simeq 3.9 \cdot 10^2\ \mathrm{Mpc} \simeq 1.2 \cdot 10^{25}\ \mathrm{m}.$$

A Newtonian circular-orbit estimate, with $2\pi f_{\mathrm{orb}} = \sqrt{\dfrac{G(m_1 + m_2)}{r^3}}$, gives

$$r \simeq \sqrt[3]{\frac{G(m_1 + m_2)}{(2\pi f_{\mathrm{orb}})^2}} \simeq 4.5 \cdot 10^5\ \mathrm{m}.$$

Substitution into the stated strain formula yields

$$h \sim 1.8 \cdot 10^{-21}, \qquad \mathrm{SNR} \sim \frac{h}{h_{\mathrm{noise}}} \sim 1.3 \cdot 10^{2}.$$

This is the order-of-magnitude SNR defined by the effective noise amplitude in the question.

Marking Scheme (20 points):

- **Question (a) Upper bound on the area increase**
  - **A1** Derivation of the upper bound on the fractional area increase **2**
  - **A2** A limiting example with equal initial masses, nearly maximal initial spins, a nearly non-rotating remnant, and negligible radiated energy **3**

- **Question (b) Final mass and radiated energy**
  - **B1** Measurement of the fractional area increase $x = \Delta\mathcal{A}/\mathcal{A}_i$ from the gray band **2**
    - *2 points for* $0.50 \leqslant x \leqslant 0.65$*;*
    - *1 point for* $0.45 \leqslant x \leqslant 0.70$ *outside that interval.*
  - **B2** Correct final black-hole mass for the adopted area increase **2**
    For reference, the inferred mass loss is about $3\,\mathfrak{M}_\odot$ *at the band centre, about* $0.6\,\mathfrak{M}_\odot$ *at* $x = 0.70$*, and about* $5.6\,\mathfrak{M}_\odot$ *at* $x = 0.45$*.*
  - **B3** Calculation of the radiated energy, $E_{\mathrm{GW}} = (m_1 + m_2 - m_f)c^2$ **1**

- **Question (c) Wave and orbital frequencies**
  - **C1** Correct evaluation of the time unit t_M in seconds **2**
  - **C2** Measurement of a gravitational-wave period from the waveform **2**
    - *2 points for* $25t_M \leqslant T_{\mathrm{GW}} \leqslant 35t_M$*;*
    - *1 point for* $20t_M \leqslant T_{\mathrm{GW}} \leqslant 50t_M$ *outside that interval.*
  - **C3** Correct orbital frequency from $f_{\mathrm{orb}} = f_{\mathrm{GW}}/2$ **1**

- **Question (d) Strain and signal-to-noise ratio**
  - **D1** Distance from Hubble's law, either directly or by rescaling the upper panel **2**
  - **D2** Estimation of the strain h **1**
    Accept $0.6 \cdot 10^{-21} \leqslant h \leqslant 6 \cdot 10^{-21}$*. A direct reading of* h *from the graph is also acceptable; in that case award full credit for D1 as well.*
  - **D3** Calculation of the signal-to-noise ratio, $\mathrm{SNR} = h/h_{\mathrm{noise}}$ **2**
    1 point if the method is correct but the final answer is wrong.

5 A Spectre Is Haunting Europa

In 1848, Karl Marx and Friedrich Engels wrote, *"A spectre is haunting Europe — the spectre of Communism."*

Suppose a translucent spectre (= ghost) wanders over the surface of Jupiter's moon Europa and sometimes appears on the side facing Earth. The ghost reflects half the light falling on it and transmits the other half. Assume diffuse reflection and transmission.

The largest telescope available at the time was the Leviathan of Parsonstown, completed in 1845, with an aperture of 1.83 m. Could an astronomer detect the spectre of communism on Europa?

Solution:

The spectre could reveal itself either as a feature on Europa's image or through a change in the moon's total brightness.

Direct observation? Near opposition, Europa is about $D = (5.2 - 1)$ au $\simeq 6.3 \cdot 10^{11}$ m from Earth. Its angular diameter is therefore

$$\theta_E \simeq \frac{2R_E}{D} = \frac{2 \times 1.56 \cdot 10^6}{6.3 \cdot 10^{11}} \text{ rad} \simeq 1.0 \text{ arcsec.}$$

For the Leviathan's aperture $d = 1.83$ m, the diffraction limit at $\lambda = 550$ nm is

$$\theta_{\text{diff}} = 1.22\frac{\lambda}{d} \simeq 0.076 \text{ arcsec.}$$

Even this ideal resolution corresponds to about 230 km on Europa. Atmospheric seeing of order 1 arcsec would blur the image on the scale of Europa's entire disc. Direct observation of a small spectre is consequently impossible with this telescope.

An unresolved object can still affect the total received light, however, so we must also estimate the photometric signal. Its size is unspecified, so we shall keep its covered area S as a parameter.

Change in brightness? First consider the light from Europa itself. Taking its geometric albedo as $A_E \simeq 0.7$ and its heliocentric distance as $r = 5.2$ au, the flux at Earth near full phase is

$$\frac{F_0}{F_{\odot,1}} = A_E \cdot \left(\frac{1 \text{ au}}{r}\right)^2 \cdot \left(\frac{R_E}{D}\right)^2 \simeq 1.59 \cdot 10^{-13},$$

where $F_{\odot,1}$ is the solar flux at 1 au in the same band. The two distance factors account for the sunlight reaching Europa and the reflected light reaching Earth. With $m_{\odot,V} = -26.74$, Pogson's law gives

$$m_E = -26.74 - 2.5 \log_{10} 1.59 \cdot 10^{-13} \simeq 5.26.$$

The telescope has collecting area $S_{\text{tel}} = \pi d^2/4 \simeq 2.63\ \text{m}^2$ and receives power $\eta S_{\text{tel}} F_0$, where η is its throughput. Europa itself is readily visible; the relevant question is how much the spectre changes this flux. As a rough scale for visual brightness comparisons, a change of $\delta m \sim (0.1 \div 0.2)$ mag corresponds to a fractional brightening

$$\epsilon_{\text{vis}} = 10^{0.4\delta m} - 1 \sim (0.1 \div 0.2).$$

The spectre both reflects sunlight and intercepts light travelling to and from the surface beneath it. To include the background consistently, let a and a_c be the diffuse reflectances of bare and covered regions, and let f be the fraction of Europa's original observed flux supplied by the covered region. For matching angular scattering laws,

$$\frac{F}{F_0} = (1-f) + f \cdot \frac{a_c}{a},$$

$$\epsilon \equiv \frac{F - F_0}{F_0} = f\left(\frac{a_c}{a} - 1\right).$$

Thus local contrast is diluted by the fraction of the disc that is covered. The common distance, aperture, and throughput factors cancel from this ratio. For an illustrative bright surface with $a \simeq 0.7$ and a passive covering, $0 \leqslant a_c \leqslant 1$ gives the generous bound $|\epsilon| \leqslant f$. Here a is a local diffuse reflectance, distinct from the geometric albedo A_E used to estimate Europa's apparent magnitude.

At opposition, a small patch at the disc centre of a uniform Lambert sphere makes the largest contribution for a given area S. The illumination and projection weighting is $\cos^2\theta$, whose integral over the visible hemisphere is $\frac{2}{3}\pi R_E^2$. Consequently,

$$f \leqslant \frac{3S}{2\pi R_E^2} \simeq 1.96 \cdot 10^{-13} \times \frac{S}{1\ \text{m}^2}.$$

For $S = 1\ \text{m}^2$, even an order-unity local contrast would therefore produce only

$$\Delta m = -2.5 \log_{10}(1+\epsilon) \simeq -1.086\epsilon,$$
$$|\Delta m| \lesssim 2 \cdot 10^{-13}\ \text{mag}.$$

This is about $5 \cdot 10^{11}$ times smaller than a 0.1 mag visual comparison.

Multiple reflections? The stated reflection and transmission fractions allow a more detailed contrast estimate. Write its reflectance and transmittance as $R = T = 1/2$. Direct reflection contributes R; transmission to the surface and back contributes $T^2 a$; each further round trip adds a factor Ra. Hence

$$a_c = R + T^2 a \sum_{n=0}^{\infty} (Ra)^n = R + \frac{T^2 a}{1 - Ra} = \frac{1}{2-a}.$$

For $a = 0.7$, this gives

$$\epsilon = \frac{9f}{91}, \qquad \Delta m = -2.5 \log_{10}\left(1 + \frac{9f}{91}\right) \simeq -0.107f \text{ mag} \quad (f \ll 1).$$

Even complete coverage would change the magnitude by only -0.1 mag in this model; small coverage produces a proportionally smaller signal. Therefore, the detection is impossible.

Marking Scheme (20 points):

A Estimation and justified comparison of angular size and aperture resolution, including seeing where relevant **4**

B Appropriate Earth–Europa and Sun–Europa distances and allowance for both inverse-square losses, or equivalent consistent flux reasoning **4**

C Estimation of the flux change, including coverage dilution, and comparison with a plausible visual threshold using Pogson's law **4**

D Correct background accounting and treatment of reflection and transmission, with clear assumptions about geometry and albedo **4**

E Distinction between imaging and photometry and identification of the controlling limitations **2**

F A conclusion supported by the estimates, with acknowledgement of the unspecified size and detection threshold **2**

6 Interstellar Cloud Computing

Astronomers observe two planetary nebulae. Model each nebula as a uniform, optically thin sphere of ionized gas, and assume both nebulae have the same temperature and composition. The volume emissivity (power emitted per unit volume) in the Hβ line is proportional to the square of the electron number density: $\varepsilon_{\mathrm{H}\beta} \propto n_e^2$.

The measured properties are shown below:

		Nebula A	**Nebula B**
Angular diameter	θ ($''$)	40	20
Hβ line flux	F (10^{-10} erg cm^{-2} s^{-1})	1.0	1.3
Ionized gas mass	$\mathfrak{M}$ ($\mathfrak{M}_\odot$)	0.28	0.10

An independent method gives the distance to Nebula A as $d_A = 1.0$ kpc. Estimate the distance to Nebula B. Ignore interstellar extinction.

Solution: Let l be the linear diameter of a nebula. The spheres have the same composition, so their electron densities scale as $n_e \propto \mathfrak{M}/l^3$. Their H$\beta$ luminosities therefore obey

$$L_{\mathrm{H}\beta} \propto \varepsilon_{\mathrm{H}\beta} l^3 \propto n_e^2 l^3 \propto \frac{\mathfrak{M}^2}{l^3}.$$

For small angular diameters, $l \simeq \theta d$, while the measured line flux is

$$F = \frac{L_{\mathrm{H}\beta}}{4\pi d^2}.$$

The same proportionality constant applies to both nebulae; hence $F \propto \dfrac{\mathfrak{M}^2}{\theta^3 d^5}$,

$$\frac{d_B}{d_A} = \left(\frac{\mathfrak{M}_B}{\mathfrak{M}_A}\right)^{2/5} \cdot \left(\frac{\theta_A}{\theta_B}\right)^{3/5} \cdot \left(\frac{F_A}{F_B}\right)^{1/5}.$$

Equivalently, $d \propto \mathfrak{M}^{2/5}\,\theta^{-3/5}\,F^{-1/5}$. This is the Shklovsky distance formula for the adopted uniform-nebula model. Substituting the table values, with identical units in each ratio, gives

$$d_B = 1.0 \text{ kpc} \times \left(\frac{0.10}{0.28}\right)^{2/5} \times \left(\frac{40}{20}\right)^{3/5} \times \left(\frac{1.0}{1.3}\right)^{1/5} \simeq 0.95 \text{ kpc}.$$

Marking Scheme (20 points):

A Use of the luminosity scaling $L \propto \varepsilon r^3$ **3**

B Use of the electron-density scaling $n_e \propto \mathfrak{M}/r^3$ **3**

C Relation between linear size, angular size, and distance **3**

D Use of the inverse-square law for the observed flux **4**

E Derivation of the distance expression through proportionalities or ratios of the known quantities **4**
Nonzero credit for E *only if the expression is dimensionally correct.*

F Correct final distance to within 0.1 kpc **3**
Otherwise, 1 point for a justified answer in the range $(0.1 \div 10)$ kpc.

7 Ulysses' Quick Visit

The Ulysses spacecraft was launched toward Jupiter in October 1990. It left Earth tangentially to Earth's orbit and then moved under the Sun's gravity until it reached Jupiter. After a passive gravity-assist maneuver, it entered a polar orbit around the Sun with eccentricity $e = 0.6$. Assume that Ulysses was at the aphelion of this orbit after its flyby of Jupiter and was heading toward the Sun's south pole.

What was the latitude of the point on Jupiter's surface that came closest to the spacecraft during the flyby? Neglect the tilt of Jupiter's rotation axis.

Solution:

Let $r_E = 1$ au, $r_J = a_J$, $q = r_J/r_E = 5.2$, and $V = \sqrt{G\mathfrak{M}_\odot/r_J}$ be Jupiter's orbital speed. At the aphelion of the new solar orbit, the spacecraft's heliocentric speed is $w = V\sqrt{1-e}$. Since this orbit is polar, its velocity there is perpendicular to Jupiter's velocity. The outgoing Jupiter-relative speed is therefore

$$U = \sqrt{V^2 + w^2} = V\sqrt{2-e}.$$

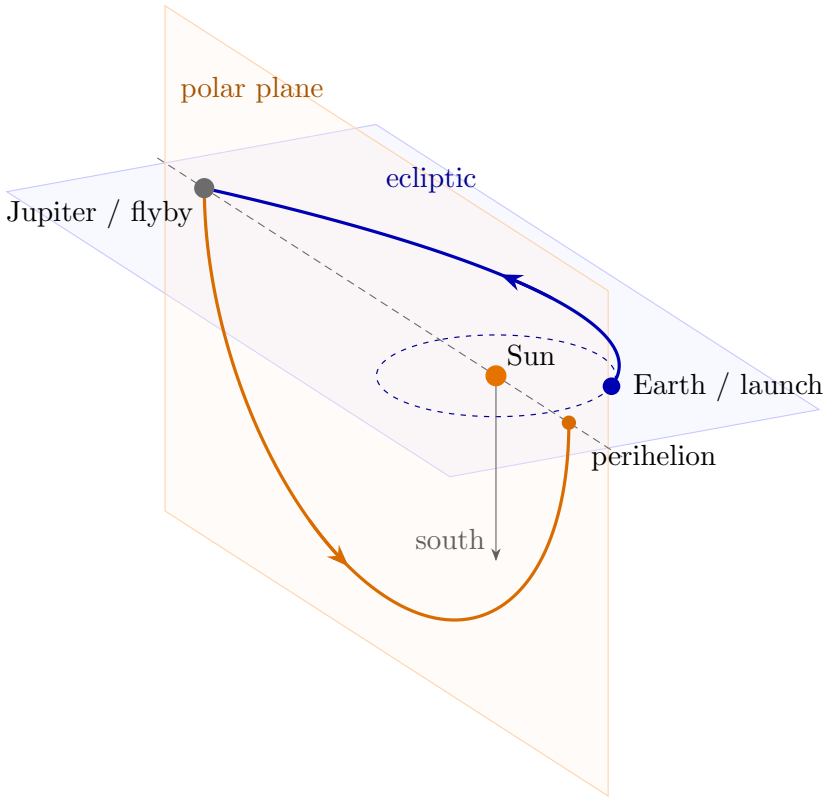


Figure 3: Schematic heliocentric path of the spacecraft. The spacecraft travels from Earth to Jupiter in the ecliptic, then follows the southern half of its polar solar orbit

Let v_0 be the spacecraft's tangential heliocentric launch speed at Earth. Just before the Jupiter encounter, let v_r and v_t be the radial and tangential components of its heliocentric velocity. Conservation of angular momentum about the Sun gives

$$r_E v_0 = r_J v_t, \text{or} v_0 = q v_t.$$

Conservation of heliocentric energy then gives

$$v_r^2 + v_t^2 = v_0^2 + 2G\mathfrak{M}_\odot \left(\frac{1}{r_J} - \frac{1}{r_E}\right) = q^2 v_t^2 - 2(q-1)V^2.$$

A passive flyby preserves the speed *relative to Jupiter*. Before the encounter this speed is $\sqrt{v_r^2 + (v_t - V)^2}$, so

$$v_r^2 + (v_t - V)^2 = U^2 = (2-e)V^2.$$

Eliminating v_r and setting $x = v_t/V$ gives

$$q^2 x^2 - 2x - (2q - 1 - e) = 0,$$

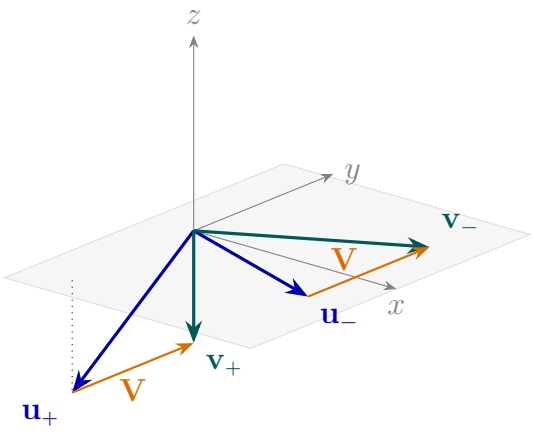
$$x = \frac{1 + \sqrt{1 + q^2(2q-1-e)}}{q^2} \simeq 0.609.$$

The other root would give $v_r^2 < 0$ and is inadmissible. Hence

$$v_t \simeq 0.609V,$$
$$v_r = V\sqrt{(2-e) - (x-1)^2} \simeq 1.117V.$$

Choose x in the outward radial direction at Jupiter, y along Jupiter's orbital motion, and z toward Jupiter's north pole. The incoming and outgoing velocities relative to Jupiter are

$$\mathbf{u}_- = (v_r,\ v_t - V,\ 0),$$
$$\mathbf{u}_+ = (0,\ -V,\ -w),$$
$$|\mathbf{u}_-| = |\mathbf{u}_+| = U.$$

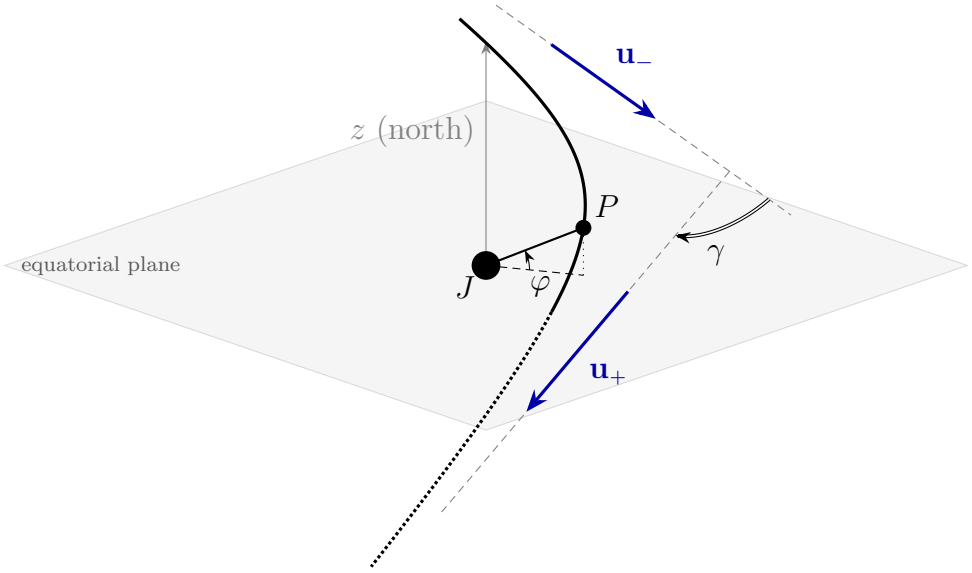


In the heliocentric frame,

$$\mathbf{v}_- = (v_r,\ v_t,\ 0),$$
$$\mathbf{v}_+ = (0, 0, -w).$$

With Jupiter's velocity $\mathbf{V} = (0,\ V,\ 0)$, both pairs satisfy $\mathbf{v}_\pm = \mathbf{u}_\pm + \mathbf{V}$. The two views below show the same three-dimensional vectors; z points north.

Geometric approach. The axis toward periapsis of a hyperbolic flyby bisects the directions of the incoming asymptotic velocity $\mathbf{u}_-$ and the reversed outgoing velocity $-\mathbf{u}_+$. It therefore points along $\mathbf{u}_- - \mathbf{u}_+$. The turning angle γ between the two asymptotic velocity directions follows from their scalar product:



$$\cos\gamma = \frac{\mathbf{u}_- \cdot \mathbf{u}_+}{U^2} = \frac{V(V - v_t)}{V^2 + w^2} = \frac{1 - x}{2 - e}, \qquad \gamma \simeq 73.8^\circ.$$

The isosceles triangle formed by $\mathbf{u}_-$ and $\mathbf{u}_+$ has base $|\mathbf{u}_- - \mathbf{u}_+| = 2U\sin(\gamma/2)$. Its base has vertical component w, because the incoming velocity has no vertical component.

Thus the latitude of periapsis is

$$\sin\varphi = \frac{w}{2U\sin(\gamma/2)} \simeq 0.445, \qquad \varphi \simeq 26.4^\circ.$$

Analytic approach. For the Jupiter-centred hyperbola, let $\mathbf{e}_h$ denote the Laplace–Runge–Lenz (eccentricity) vector:

$$\mathbf{e}_h = \frac{\mathbf{u} \times \mathbf{h}}{G\mathfrak{M}_J} - \hat{\mathbf{r}}, \qquad \mathbf{h} = \mathbf{r} \times \mathbf{u}.$$

This vector is defined for any Kepler orbit and points toward periapsis whenever the orbit is noncircular; for a circle it vanishes and periapsis has no unique direction. Far before and far after the encounter, $\hat{\mathbf{r}}$ tends to $-\mathbf{u}_-/U$ and $\mathbf{u}_+/U$, respectively. Taking the scalar product of $\mathbf{e}_h$ with each asymptotic velocity therefore gives

$$\mathbf{e}_h \cdot \mathbf{u}_- = U, \qquad \mathbf{e}_h \cdot \mathbf{u}_+ = -U.$$

Since $\mathbf{e}_h$ lies in the plane of the encounter, these equations and $|\mathbf{u}_-| = |\mathbf{u}_+|$ imply

$$\hat{\mathbf{r}}_p = \frac{\mathbf{u}_- - \mathbf{u}_+}{|\mathbf{u}_- - \mathbf{u}_+|} = \frac{(v_r,\ v_t,\ w)}{\sqrt{v_r^2 + v_t^2 + w^2}}.$$

Taking the ratio of its vertical component to the length of its projection on the ecliptic gives an independent calculation:

$$\tan\varphi = \frac{w}{\sqrt{v_r^2 + v_t^2}} = \frac{\sqrt{1-e}}{\sqrt{(2-e) - (x-1)^2 + x^2}} \simeq 0.497, \qquad \varphi \simeq 26.4^\circ \text{ N}.$$

The latitude is northern because the outgoing spacecraft moves toward the Sun's south pole.

Marking Scheme (20 points):

- **Approach velocity in Jupiter's frame**
  - **A1** Use of conservation of heliocentric angular momentum **2**
  - **A2** Use of conservation of heliocentric energy **2**
  - **A3** Allowance for Jupiter's orbital velocity in the change of reference frame **1**
  - **A4** Correct radial and tangential approach velocities in Jupiter's frame, numerically or in terms of known quantities **2**
- **Periapsis direction and latitude**
  - **B1** Decomposition of the incoming Jupiter-relative velocity; alternatively, the general expression for the Laplace–Runge–Lenz vector **3**
  - **B2** Calculation of the hyperbolic turning angle; alternatively, evaluation of the Laplace–Runge–Lenz vector on the corresponding turning geometry **4**
  - **B3** Derivation of the periapsis latitude **3**
  - **C** Correct numerical periapsis latitude, approximately 26.4° N, to within 2° **3**

8 Times Square

The table gives the present-day heliocentric positions and velocity components of two stars:

α	δ	r (pc)	μ_α (mas yr^{-1})	μ_δ (mas yr^{-1})	v_r (km s^{-1})
$14^{\mathrm{h}}\,00^{\mathrm{m}}\,00.000^{\mathrm{s}}$	$0^\circ\,00'\,00''$	3.250	-90	0	0
$09^{\mathrm{h}}\,00^{\mathrm{m}}\,00.000^{\mathrm{s}}$	$20^\circ\,00'\,00''$	4.750	0	-180	0

Find how close the stars will come to each other and how many years from now that will happen. Assume each star's heliocentric velocity is constant in space.

Solution:

Take the x axis toward right ascension 0, the z axis toward the north celestial pole, and the y axis to complete a right-handed system. At the present epoch, a star at distance r has position

$$\mathbf{r} = r(\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta).$$

The proper motions are angular velocities in milliarcseconds per year.

With $k = 4.848 \cdot 10^{-9}$ rad mas^{-1}, differentiation gives the present velocity

$$\mathbf{v} = v_r\mathbf{e}_r + rk\big(\mu_\alpha\cos\delta\,\mathbf{e}_\alpha + \mu_\delta\,\mathbf{e}_\delta\big),$$

where

$$\mathbf{e}_\alpha = (-\sin\alpha,\ \cos\alpha,\ 0),$$
$$\mathbf{e}_\delta = (-\sin\delta\cos\alpha,\ -\sin\delta\sin\alpha,\ \cos\delta).$$

Both radial velocities vanish.

Note: accounting for the $\cos\delta$ *factor for* μ_α *is inconsequential, since the only non-zero element corresponds to* $\delta = 0$.

The right ascensions are $14^{\mathrm{h}} = 210^\circ$ and $9^{\mathrm{h}} = 135^\circ$. With positions in pc and velocities in 10^{-6} pc yr^{-1}, this gives

$$\mathbf{r}_1 = (-2.815,\ -1.625,\ 0), \qquad \mathbf{v}_1 = (-0.709,\ 1.228,\ 0),$$
$$\mathbf{r}_2 = (-3.156,\ 3.156,\ 1.625), \qquad \mathbf{v}_2 = (-1.002,\ 1.002,\ -3.895).$$

The relative position and velocity are therefore approximately

$$\mathbf{R}_0 = \mathbf{r}_1 - \mathbf{r}_2 = (0.342,\ -4.781,\ -1.625)\ \text{pc},$$
$$\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2 = (0.293,\ 0.226,\ 3.895)\cdot 10^{-6}\ \text{pc yr}^{-1}.$$

Closest approach. Under the stated straight-line model, $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{V}t$. Minimizing the squared separation gives

$$d^2(t) = |\mathbf{R}_0 + \mathbf{V}t|^2 = |\mathbf{R}_0|^2 + 2t\,\mathbf{R}_0 \cdot \mathbf{V} + t^2|\mathbf{V}|^2,$$

$$0 = \frac{d}{dt}d^2(t) = 2\mathbf{R}_0 \cdot \mathbf{V} + 2t_{\min}|\mathbf{V}|^2,$$

$$t_{\min} = -\frac{\mathbf{R}_0 \cdot \mathbf{V}}{|\mathbf{V}|^2} \simeq 4.77 \cdot 10^5 \text{ yr}.$$

The same result follows geometrically without differentiation. At closest approach, the separation vector is orthogonal to the relative velocity. Hence

$$\mathbf{R}(t_{\min}) \cdot \mathbf{V} = (\mathbf{R}_0 + \mathbf{V}t_{\min}) \cdot \mathbf{V} = 0, \qquad t_{\min} = -\frac{\mathbf{R}_0 \cdot \mathbf{V}}{|\mathbf{V}|^2}.$$

Substituting this time into the separation gives

$$d_{\min} = \sqrt{|\mathbf{R}_0|^2 - \frac{(\mathbf{R}_0 \cdot \mathbf{V})^2}{|\mathbf{V}|^2}} \simeq 4.70 \text{ pc}.$$

Marking Scheme (20 points):

- **Position vectors**
  - Correct position-vector formula **1**
  - Correct numerical position vector for the first star **2**
  - Correct numerical position vector for the second star **2**
  - **A Subtotal 5**
- **Velocity vectors**
  - Correct velocity-vector formula **1**
  - Correct numerical velocity vector for the first star **2**
  - Correct numerical velocity vector for the second star **2**
  - **B Subtotal 5**
  - *If only tangential speed magnitudes are calculated, 3 points max.*
- **Closest approach**
  - Closest-approach condition: perpendicular relative position and velocity, or differentiation of the separation **2**
  - Correct relative position vector **2**
  - Correct relative velocity vector **2**
  - Correct time of closest approach **2**
  - Correct minimum physical separation **2**
  - **C Subtotal 10**

Angular-separation approach (maximum 10 points):

Minimization of angular separation does not determine the minimum physical separation. Therefore, only partial credit is awarded for this approach.

A Calculation of the present distances **1**
B Calculation of the speeds **3**
C Correct expression for the angular separation + correct minimization ..... **3 + 3**

9 The Hitchhiker's Guide to the Galactic Center

Arthur Dent and Ford Prefect are looking at a radio image of the Milky Way's center. Draw the Galactic equator on the image *as accurately as possible.* Mark the direction toward the North Galactic Pole with an arrow. Justify your drawing.

Hint. The source marked Tornado (SNR) has galactic latitude $b \simeq -0.1^\circ$.

Solution:

The numbers in a supernova remnant's name give its approximate Galactic longitude l and latitude b, in degrees. Near the Galactic center, $l = 359.1^\circ$ and 359.0° can be read as -0.9° and -1.0°. The latitudes in these names and the given Tornado latitude are rounded to 0.1°, so their true values may differ by 0.05°.

Draw an *annulus* around each remnant at the image's angular scale. Its inner and outer radii, in degrees, are

SNR	0.9+0.1	359.1–0.5	359.0–0.9	Tornado
$[r_{\min}, r_{\max}]$	$[0.05, 0.15]$	$[0.45, 0.55]$	$[0.85, 0.95]$	$[0.05, 0.15]$

SNR 0.3+0.0 is too diffuse here to locate its center reliably, so we omit it.

Since $|b|$ is the shortest angular distance to the Galactic equator, the equator must be tangent to some circle inside each remaining annulus, passing on the positive-b side of the negative-b centers and on the negative-b side of SNR 0.9+0.1. This selects the diagonal running from upper left to lower right.

The positive-b side contains SNR 0.9+0.1, so the direction toward the North Galactic Pole is perpendicular to the equator, toward the upper right of the image.

Note on the Galactic Center. The central label Sgr A denotes *an extended radio complex,* not the compact source Sgr A*. The latter is at $b \simeq -0.046^\circ$, so the Galactic equator lies slightly to its north. The Sun is located slightly above the central plane of the Galaxy's disk.

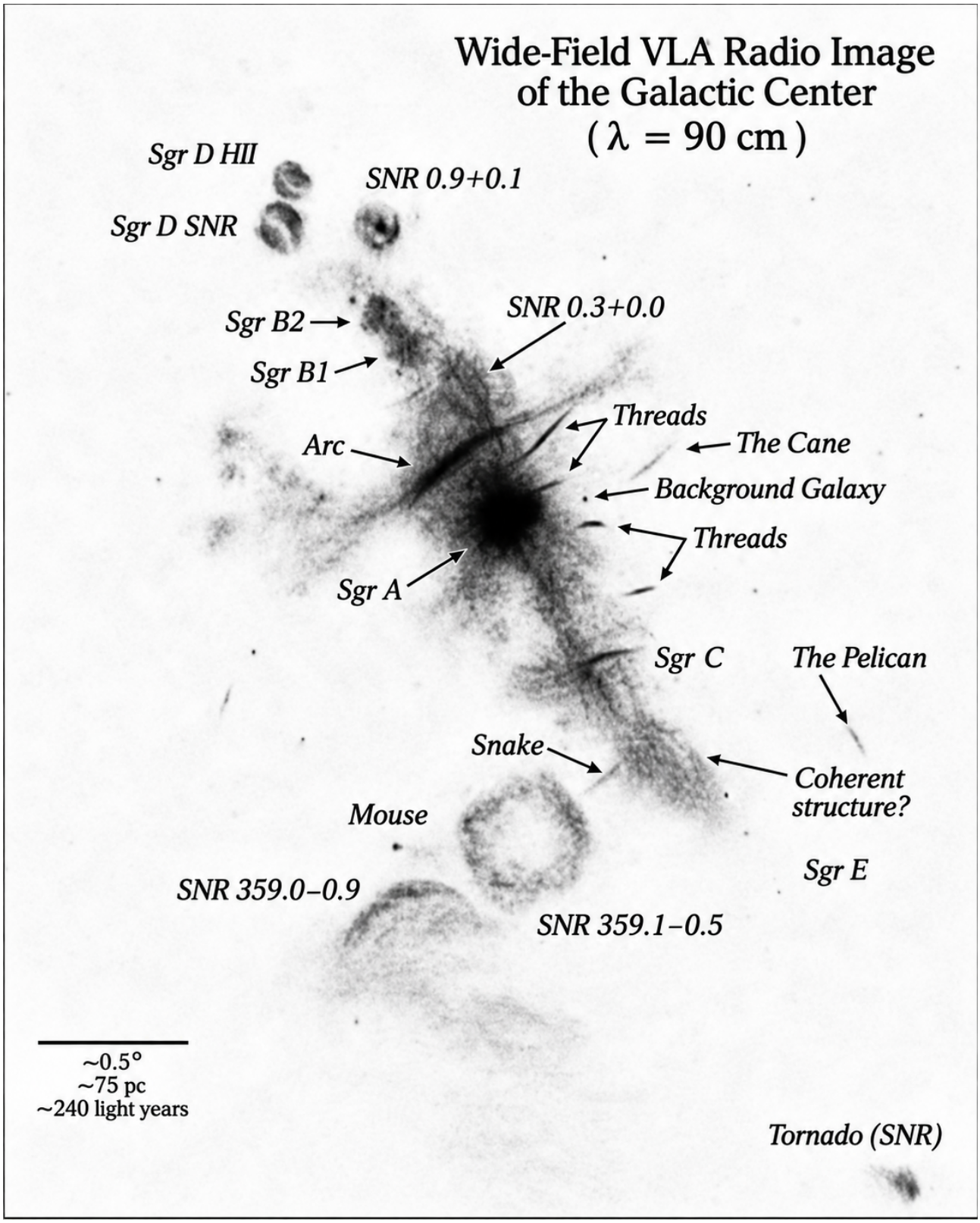


Figure 4: A 90 cm radio image of the Galactic center region by the Very Large Array. Credit: NRAO/AUI/NSF and N. E. Kassim, Naval Research Laboratory. Black-and-white version for printing and drawing

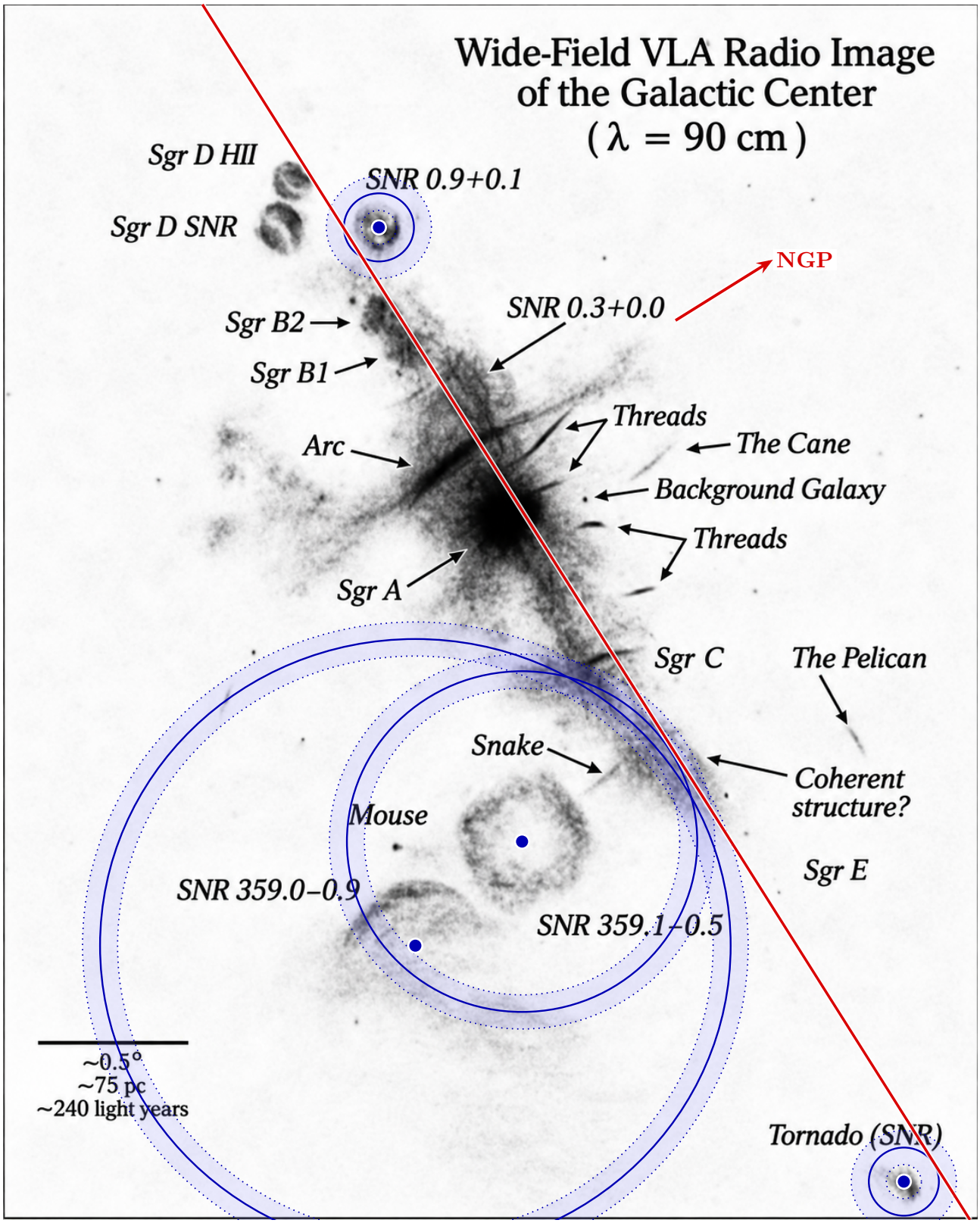


Figure 5: The Galactic equator on the radio image ($b = 0°$). The blue bands show the radii allowed by rounding each SNR's latitude, with solid circles at the nominal radii. The red line marks $b = 0°$; the arrow points toward the North Galactic Pole ($b > 0°$)

Marking Scheme (20 points):

- **Construction of the Galactic equator**

  Roughly correct direction without using SNRs 2

  A line through the Galactic centre (GC), tangent to one SNR latitude circle on the wrong side, with no additional circles 3

  A line through the GC and an SNR centre, with no additional circles 4

  A line through the GC, tangent to one SNR latitude circle on the correct side, with no additional circles 6

  Use of two latitude circles, one for the Tornado and one for another SNR 12

  Use of three or more latitude circles 16

  These scores are alternatives and are not added together.

  A Maximum 16

- **Direction to the North Galactic Pole**

  Perpendicularity of the North Galactic Pole direction to the Galactic equator ... **2**

  Correct sense of the North Galactic Pole direction **2**

  B Subtotal 4

10 You Spin Me Round

Radio telescopes observe an isolated pulsar, a rapidly rotating neutron star with a spin period of $P = 0.1$ s. After several years of observations, its period derivative is measured to be $\dot{P} = 10^{-15}$ (in units of s/s), meaning its period increases by 10^{-15} seconds each second.

The neutron star has mass $\mathfrak{M} \approx 1.4\mathfrak{M}_\odot$ and radius $R \approx 10$ km. Model the star as a uniform sphere.

The pulsar loses rotational energy through magnetic-dipole radiation. The power radiated by a rotating magnetic dipole with magnetic moment μ and angular velocity Ω is

$$\dot{E}_{\rm md} = \frac{2}{3c^3}\mu^2\Omega^4 \sin^2\alpha,$$

where α is the angle between the magnetic and rotation axes, and c is the speed of light. Assume that the magnetic moment, the angle α, and the moment of inertia remain constant.

a) Show that magnetic-dipole braking implies $P\dot{P} = \text{const}$.

b) Estimate the neutron star's moment of inertia I.

c) Find the positive rate of rotational energy loss, $\dot{E}_{\rm rot} \equiv -dE_{\rm rot}/dt$.

d) Estimate the pulsar's age τ, assuming it was born with a period $P_0 \ll P$.

Solution:

a) The rotational energy is $E_{\rm rot} = I\Omega^2/2$, where $\Omega = 2\pi/P$. The radiated power is positive, whereas the star's rotational energy decreases:

$$-I\Omega\dot{\Omega} = \dot{E}_{\rm md} = K\Omega^4, \qquad K \equiv \frac{2\mu^2\sin^2\alpha}{3c^3}.$$

Since $\dot{\Omega} = -2\pi\dot{P}/P^2$, this becomes

$$P\dot{P} = \frac{4\pi^2 K}{I} = \frac{8\pi^2\mu^2\sin^2\alpha}{3Ic^3} = \text{const}.$$

The last equality follows from the assumed constant I, μ, and α.

b) For a uniform sphere,

$$I = \frac{2}{5}\mathfrak{M}R^2 \simeq \frac{2}{5} \times (1.4 \times 1.99 \cdot 10^{30}\ \text{kg}) \times (10^4\ \text{m})^2 \simeq 1.1 \cdot 10^{38}\ \text{kg m}^2.$$

c) Differentiating $E_{\rm rot} = 2\pi^2 I/P^2$ gives the positive rate of loss

$$\dot{E}_{\rm rot} = -\frac{dE_{\rm rot}}{dt} = \frac{4\pi^2 I\dot{P}}{P^3} \simeq 4.4 \cdot 10^{27}\ \text{W}.$$

d) Writing the constant from part (a) as $C = P\dot{P}$, we have $P\,dP = C\,dt$. Integration from birth to the present gives

$$\int_{P_0}^{P} \tilde{P}\,d\tilde{P} = \int_{0}^{\tau} C\,dt,$$

$$\tau = \frac{P^2 - P_0^2}{2C} = \frac{P}{2\dot{P}}\left[1 - \left(\frac{P_0}{P}\right)^2\right].$$

Since $P_0 \ll P$,

$$\tau \simeq \frac{P}{2\dot{P}} \simeq \frac{0.1\text{ s}}{2 \cdot 10^{-15}} = 5.0 \cdot 10^{13}\text{ s} \simeq 1.6 \cdot 10^{6}\text{ yr}.$$

Marking Scheme (20 points):

- **Question (a) Magnetic-dipole braking**
  - Derivation of the magnetic-dipole braking relation **4**
  - Correct relation $P\dot{P} = \text{const}$ **2**
  - **A Subtotal** **6**
- **Question (b) Moment of inertia**
  - Use of the correct moment-of-inertia formula for a uniform sphere **4**
  - Deduct 2 points for an error in the numerical coefficient.
  - Correct numerical moment of inertia **2**
  - **B Subtotal** **6**
- **Question (c) Rotational energy loss**
  - Use of the correct formula for the positive rate of rotational energy loss **2**
  - Correct numerical energy-loss rate **2**
  - **C Subtotal** **4**
- **Question (d) Pulsar age**
  - Use of the correct age formula for $P_0 \ll P$ **2**
  - Correct numerical age **2**
  - **D Subtotal** **4**

Constants

Universal

Gravitational constant	$G = 6.67 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$
Speed of light	$c = 3.00 \cdot 10^{8}$ m/s
Planck constant	$h = 6.63 \cdot 10^{-34}$ J · s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J/K
Gas constant	$\mathfrak{R} = 8.314$ J/(mol · K)
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg

Astronomical

Astronomical unit	1 au $= 149.6 \cdot 10^{6}$ km
Parsec	1 pc $= 206\,265$ au
Hubble constant	$H_0 = 70$ (km/s)/Mpc

Earth

Radius	$R_\oplus = 6371$ km
Mass	$\mathfrak{M}_\oplus = 5.97 \cdot 10^{24}$ kg
Obliquity of ecliptic	$\varepsilon = 23.4^\circ$
Surface gravity	$g = 9.81$ m/s^2
Orbital period	$T_\oplus = 365.26$ days
Orbital eccentricity	$e_\oplus = 0.0167$

Jupiter

Radius	$R_J = 6.99 \cdot 10^{4}$ km
Mass	$\mathfrak{M}_J = 1.90 \cdot 10^{27}$ kg
Orbital radius	$a_J = 5.20$ au
Orbital period	$T_J = 11.86$ yr

Europa

Radius	$R_E = 1.56 \cdot 10^{3}$ km
Albedo	$A_E = 0.7$ (approx.)

Sun

Radius	$R_\odot = 6.96 \cdot 10^{5}$ km
Mass	$\mathfrak{M}_\odot = 1.99 \cdot 10^{30}$ kg
Absolute magnitude	$M_\odot = 4.74^{\mathrm{m}}$ (bol.)
Effective temperature	$T_\odot = 5.8 \cdot 10^{3}$ K
Luminosity	$L_\odot = 3.828 \cdot 10^{26}$ W

Emission constants

Stefan–Boltzmann	$\sigma = 5.67 \cdot 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2\cdot\mathrm{K}^4}$
Wien's displacement	$b = 2898\ \mu\mathrm{m} \cdot \mathrm{K}$